The Mathematical Education of Teachers II

The following is a summary of key points from the Conference Board of the Mathematical Sciences (CBMS) 2012 report titled *The Mathematical Education of Teachers II* (MET II). MET II updates, reiterates, and elaborates themes of the 2001 CBMS publication *The Mathematical Education of Teachers* (MET I). The complete MET II report (100 pages including references) is available online at:
http://www.cbmsweb.org/MET2/met2.pdf
The complete MET II report also includes (a) an appendix that overviews the CCSS mathematics content for grades K-12 and (b) an appendix that presents the eight CCSS Standards for Mathematical Practice.

Although much of the wording of this summary is taken directly from the text of MET II, compacting and paraphrasing have been used used to reduce the volume of text. The extent of the mathematical vocabulary of the report also has been reduced in light of the range of audiences to which it is directed. The Preface of the report states that its "primary audiences are faculty who teach in mathematics or statistics departments and their colleagues in colleges of education who have primary responsibility for the mathematical education of teachers." It goes on to state that it also is "intended to inform educational administrators and policy-makers at the national, state, school-district, and collegiate levels." However, the content-specific language used to refer to mathematics courses (and topics within them) -- especially in Chapter 6 -- could be a solid barrier to the latter audiences. In an effort to keep this summary accessible to the broader readership, mathematical details have been minimized.

Preface

The recommendations presented in MET II are intended to update MET I's recommendations regarding the mathematical preparation and continuing professional development of K-12 teachers. In so doing, it revisits themes from MET I that include the observations that there is intellectual substance in school mathematics and that the mathematical knowledge needed for teaching differs from other professions.

In seeking to determine what important mathematics teachers should know, the report turns to identifying the mathematics that teachers will be expected to teach. Noting that most states have adopted the Common Core State Standards for Mathematics (CCSS), MET II focuses on enabling teachers to teach that mathematics. To accomplish that task, MET II calls for a partnership between mathematics educators --that is, faculty within colleges of education, mathematics departments, or other academic who are responsible for the pedagogical education of mathematics teachers -- and mathematicians and statisticians -- that is, faculty of mathematics and statistics departments at two- and four-year collegiate institutions who teach mathematics and statistics courses taken by prospective and practicing teachers. Moreover, it asserts that the preparation and professional development for teachers can provide mathematicians and statisticians with "genuinely interesting intellectual experiences, affording the opportunity to 'think deeply about simple things,' and to make connections between the undergraduate courses that they teach and K–12 mathematics."

Chapter 1
School Mathematics and Teachers' Mathematics

Chapter 1 describes the mathematical issues that underlie the recommendations of MET II, including the structure and content of the CCSS. MET II also uses the phrase "habits of mind" in describing ways of thinking about mathematics that are found in the CCSS Standards for Mathematical Practice. The report asserts that, at every level of K-12 education, there is intellectually demanding mathematics to be learned that is widely used. Teachers at any one of three grade groupings -- K-5, 6-8, and 9-12 -- "need to know how the mathematics they teach is connected with that of prior and later grades." For example, teachers need to understand such things as the need for definitions and the consequences of alternative definitions (e.g., whether a trapezoid is defined as a quadrilateral with ***exactly*** one pair of parallel sides or ***at least*** one pair parallel sides.). All teachers of mathematics need to be able to detect flaws in students arguments and to help students the nature of those errors. There must be high standards for entry into the teaching profession, and there must be continual improvement in mathematical knowledge and teaching skills through "regular interactions among teachers, mathematicians, and mathematics education faculty in creating and analyzing lessons, textbooks, and curriculum documents... ." The

mathematical education of teacher must become of central concern to mathematicians and collegiate mathematics departments.

The level of mathematical knowledge needed by teachers at each of the three grade levels presents a particular challenge. Future teachers of elementary school children may enter college with only superficial knowledge of K-12 mathematics. Moreover, once certified, it is rare for K-6 teachers to have "any sustained professional development centered on mathematics." Middle school teachers frequently have the same mathematical preparation as elementary generalists, a practice that must be replaced by the requirement that middle school teachers of mathematics specialize in the study of mathematics. And although high school teachers of mathematics frequently major in mathematics, the content of the courses that they take often does not provide the advanced mathematical perspectives needed for teaching in high school. Mathematics courses for high school teachers should prepare them to "continue to learn new mathematical content and to deepen their understanding of familiar topics."

Chapter 2
The Mathematical Education of Teachers: Traditions, Research, Current Context

Chapter 2 summarizes empirical findings regarding issues that underlie the recommendations of MET II; it connects those recommendations with the current context of education. Its three sections address five issues: (1) mathematician's roles in teacher education; (2) beliefs about mathematics and their influences on learning; (3) practices in teaching mathematics and their influence on learning; (4) research related to the relationship between teacher effectiveness and mathematical knowledge; and (5) the current context affecting teacher preparation and supply/demand.

Traditions, Beliefs, and Practices

Mathematicians' concern for school mathematics and the education of teachers stretches back into the eighteenth century. Although mathematicians' engagement declined in the twentieth century, there are notable exceptions. [NOTE: For instance, mathematicians contributed to the development of school curriculum materials published by such groups as Developing Mathematical Processes, School Mathematics Study Group, University of Illinois Committee on School Mathematics. Those projects included programs to prepare teachers to use the materials.]

A 1992 book by Alan Schoenfeld reported the following two student beliefs about mathematics (specifically, geometry) as those beliefs were revealed in the classroom behavior of the students.

- "Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
- ”Ordinary students cannot expect to understand mathematics: they expect simply to memorize it and apply what they have learned mechanically and without understanding."

The source of such beliefs is traced back to the pedagogy of the "rule method" -- simply memorize a rule and practice using -- that was promoted by textbooks from colonial times into the 1820s. Mathematical "facts" were memorized in isolation from one another; no attempt was made to connect them by employing the structural principles of mathematical systems (e.g., commutative and distributive principles). These practices had a rebirth in the back-to-the-basics movement of the 1980s. Moreover, research identified corresponding counterproductive beliefs regarding the interactions of students and teachers in the classroom.

- "Doing mathematics means following the rules laid down by the teacher.
- "Knowing mathematics means remembering and applying the correct rule when the teacher asks a question.
- "Mathematical truth is determined when the answer is ratified by the teacher."

Also, research studies found "far fewer occurrences of deductive reasoning in U.S. mathematics classes than in classrooms from countries whose students score well on international tests."

Teacher Effectiveness and Mathematical Knowledge

Research provides evidence that the preparation and professional development of teachers should be "tailored to the work of teaching." MET II summarizes research on teacher effectiveness -- the effect that a teacher has on students' learning -- as it is related to four sources of teaching-focused knowledge.

1. Mathematics courses and certification: Studies have focused on the mathematics courses associated with the preparation of middle school and high school teachers of mathematics. In general, more mathematics is associated with better student performance, but the effect sizes are small and inconsistent.
2. Mathematical knowledge for teaching: Beginning in the 1980s, research studies turned to investigating specific kinds of mathematical knowledge that are *relevant* to teaching as, for instance, when students raise mathematical questions. Called "mathematics for teaching," it includes the mathematical knowledge needed to respond to a question such as "Why can't you divide by zero?" or "When you divide fractions, how come you 'invert and multiply'?"
3. Curriculum-specific professional development: Recent studies show that teacher workshops that provided sustained examination of curriculum-specific mathematics and how to teach it were associated with improved performance of students on mathematics tests.
4. Teaching-learning paths: CCSS is structured by what are called "learning pathways" or "learning trajectories" in which the learning of a mathematical concept is not an event but a multi-grade continuum. Student progress along those trajectories is enhanced by sustained and specific teacher professional development that helps teachers "perceive the elements of a given concept or skill, and mathematical stepping-stones in their development."

Current context

The context of education has experienced significant changes since the CBMS published MET I in 2001.

- Secondary schools are experiencing difficulties in filling positions for teachers of mathematics.
- A identified factor in teacher turnover is the provision of content-based professional development. Schools providing professional development that was perceived to be useful had substantially lower rates of turnover.
- About 40% of teachers have entered the field through an "alternative pathway." Survey data indicates that teachers entering with a degree in mathematics but "little or no pedagogical preparation" left the field at twice the rate of those with a mathematics degree and "more comprehensive pedagogical preparation."
- Mathematics departments are designing and providing specially designed courses or special sections of mathematics courses for K-8 teachers.
- The preparation and continuous development of mathematics teachers has become a collaborative effort. "Mathematicians have expanded their involvement in mathematics education, forming partnerships with mathematics education researchers, education officials, and teachers in new kinds of programs."

Chapter 3
Recommendations: Mathematics for Teachers; Roles for Mathematicians

Chapter 3 targets college/university department chairs, educational administrators, and policy-makers at the national, state, school-district, and collegiate levels. It present 4 recommendations for strengthening the mathematical education of teachers and 2 recommendations regarding the roles of mathematicians and statisticians in accomplishing that strengthening.

Recommendations: Mathematics for teachers

1. Courses for prospective teachers of mathematics should develop solid understanding of the mathematics that those teachers will teach.
2. Coursework should allow time for prospective teachers to engage in reasoning, explaining, and making sense of the mathematics that they will teach. While he number of semester hours of coursework varies by school level (elementary, middle grades, high school), each level should include courses especially designed for teachers.
3. Throughout their careers, teachers should have professional development opportunities, school- and district-based, for continued expansion of their mathematical knowledge. This is of increased importance due to widespread adoption of CCSS.

4. All pre-service and professional development coursework for mathematics teachers should develop the habits of mind of a mathematical thinker and problem solver. In order to develop these abilities in their students [as called for in the CCSS in terms of its Standards for Mathematical Practice], teachers must experience them in their own mathematical education.

Recommendations: Roles for Mathematicians in Teacher Education

- Teacher education must be recognized as an important part of the mission of the mathematics and statistics departments in any institution that offers pre-service or professional development courses for teachers; that work should be undertaken in collaboration with the mathematics education faculty of the institution. The departments should allocate commensurate resources for designing and staffing courses for teachers. Also, they should encourage and reward department members who participate in the preparation and professional development of teachers and who become involved with local schools or districts.
- Mathematicians should recognize the need for improving mathematics teaching at all levels. Mathematicians should support the growth of professionalism of mathematics teachers by ensuring that the knowledge PreK-12 mathematics teachers is sufficient at the time of initial certification, by encouraging mathematics teachers' continual improvement, and by joining with teachers at different levels to learn with and from each other.

Chapter 4
Elementary Teachers

Chapters 4 provides recommendations to those engaged in the preparation or professional development of elementary grades teachers (which MET II defines to be grades K-5). Overall, the recommendations call for beginning K-5 teachers to "study in depth, and from a teachers perspective, the vast majority of K-5 mathematics, its connections to prekindergarten mathematics, and its connections to grades 6-8 mathematics." Elementary grades teachers also should complete courses in mathematical pedagogy. The mathematics to be learned and, later, taught is that described in the CCSS. Chapter 4 is in two sections. The first section presents brief sketches of how CCSS mathematics might be treated in courses in pre-service and professional development courses. The second section suggests how that mathematics might be organized in courses, programs, or seminars for teachers.

Section 1: Essential Mathematical Ideas for Grades K-5 Teachers

This section lists the essential ideas of mathematics in the K-5 domain and attends to how they build toward those of grades 6-8. MET II describes each of the domains named below in terms of key topics to be addressed; a selected few of those topics are given here. Each domain also is accompanied by a list of related activities that could be used in teacher preparation or professional development. Those lists (not included below) relate the activities to specific Standards for Mathematical Practice.

- Counting and Cardinality (Kindergarten): learning to count, including the distinction between counting as a list of numbers in order and counting to determine a number of objects.
- Operations and Algebraic Thinking (Kindergarten-Grade 5): solving problems by addition, subtraction, multiplication, and division, and meanings of the operations illustrated by these problem types; understanding the equal sign as meaning “the same amount as” rather than a “calculate the answer” symbol.
- Number and Operations in Base Ten (Kindergarten-Grade 5): understanding how the base-ten place value system relies on repeated bundling in groups of ten; understanding how base-ten computation methods for addition, subtraction, multiplication, and division rely on decomposing numbers represented in base ten; extending the base ten system to decimals.
- Measurement and Data (Kindergarten-Grade 5): the general principles of measurement, the process of iterations, and the central role of units; that concept that measurement is comparison with a unit and how the size of a unit affects measurements.
- Geometry (Kindergarten-Grade 5): understanding the geometric concepts of angle, parallel, and perpendicular, and using them in describing and defining shapes; classifying shapes into categories and explaining relationships among the categories.

Section 2: The Preparation and Professional Development of K-5 Teachers

MET II recommends that programs designed to prepare K-5 teachers should include 12 semester-hours focused on the study of CCSS mathematics for grades K-5 and related aspects of 6-8. Since the mathematics of elementary school can be studied repeatedly with increasing depth and attention to detail, practicing teachers will benefit from deeper study of the very topics they studied in their pre-service preparation programs. Insofar as possible, courses should blend the study of content and pedagogy. MET II stresses that familiar mathematics courses offered by departments of mathematics (e.g., college algebra, liberal arts mathematics) *"are not an appropriate substitute for the study of mathematics for elementary teachers*."

Professional development in the study of mathematics by K-5 teachers may take many forms.

- A group of teachers might focus on the study of one topic for an extended period of time.
- A group of teachers with multi-grade assignments in one school might study how related topics develop across grade levels.
- A group of teachers might observe -- and then discuss -- demonstration lessons taught by a mathematics specialist.
- Professional development can take place at a school location (during or after school hours) or on a college campus.

Teacher preparation and professional development can "provide opportunities to *do* mathematics and to develop mathematical habits of mind. Specifically, the education of teachers should include opportunities "to reason abstractly and quantitatively, to construct viable arguments, to listen carefully to other people's reasoning, and to discuss and critique it." Teachers should have opportunities to engage in mathematical modeling -- that is, focusing on the mathematical aspects of a situation and formulating them in mathematical terms. They should learn to use mathematical terminology and notation correctly.

Teachers must experience and learn to use an expanding array of education technology tools. Traditional teaching tools such as blackboards, base-ten blocks, tiles, and counters still have a role in instruction. However, teacher preparation and professional development also must provide opportunities for teachers to use whiteboards, tablets, applets, apps, web sites, and multimedia resources in their own learning. Those experiences can provide teachers with awareness of the strengths and limitations of the technology tools and how such tools can be effectively employed in problem solving.

Chapter 4 closes with subsections that address the roles and preparation of elementary school mathematics specialists, early childhood teachers, and teachers of special populations of children.

Chapter 5
Middle Grades Teachers

Chapter 5 provides recommendations to those engaged in the preparation or professional development of middle grades teachers (which MET II defines to be grades 6-8). The mathematics to be learned and, later, taught is that described in the CCSS. Chapter 5 is in two sections. The first section presents brief sketches of how that mathematics might be treated in courses in pre-service and professional development courses. The second section suggests how that mathematics might be organized in courses, programs, or seminars for teachers.

Section 1: Essential Mathematical Ideas for Teachers of Grades 6-8

This section lists the essential ideas of the CCSS mathematics domains for grades 6-8 and attends to how they build toward those of grades 9-12. Each domain named below is described in terms of key topics to be addressed; a selected few of those topics are given here. Each domain also is accompanied by a list of related activities that could be used in teacher preparation or professional development. Those lists (not included below) relate the activities to specific Standards for Mathematical Practice.

- Ratio and Proportional Relationships (Grades 6-7): reasoning about how quantities vary together by using tables, tape diagrams (a drawing that looks like a segment of tape and that is used to illustrate number relationships) and double number lines (two number lines, one above the other, displaying related information); viewing the concept of proportional relationship as an example of a linear relationship.

- The Number System (Grades 6-8: understanding and explaining methods of calculating using area models, tape diagrams, and double number lines and relationships of quantities expressed by equations; using field axioms to explain operations with rational real numbers.
- Expressions and Equations (Grades 6-8): parsing numerical and algebraic expressions into their component parts and interpreting those components in terms of a context; examining lines of reasoning used to solve equations and systems of equations.
- Functions (Grade 8): examining functional relationships that are represented using tables, graphs, equations, and verbal descriptions; examining the types of real-world relationships that can be modeled by linear, inverse proportional, quadratic, and exponential functions.
- Geometry (Grades 6-8): deriving selected area formulas; giving multiple justifications of the Pythagorean Theorem and applying its converse; understanding congruence and similarity in terms of transformations, rotations, reflections, and dilations.
- Statistics and Probability (Grades 6-8): understanding ways to summarize, describe, and compare distributions of numerical data in terms of shape, center, and spread; understanding statistical variability and its sources; understanding the role of randomness in statistical inference.

Section 2: The Preparation and Professional Development of Teachers of Grades 6-8

Middle school teachers, in addition to teaching the CCSS mathematics of grades 6-8, must be aware of the mathematics students studied in grades K-5 and must be able to anticipate and make connections with the mathematics to be learned in grades 9-12 and beyond. Therefore, teachers of the middle grades need to specialize in mathematics. In addition to two methods courses that address the teaching and learning of middle school mathematics, prospective middle school teachers should complete 24 semester hours of mathematics, at least 15 semester hours of which should be courses specifically designed for middle grades teachers. The CCSS-related content of those 15 semester hours of mathematics are indicated by the following titles. (The MET II document fleshes out their content.)

- Number and operations (6 semester hours)
- Geometry and measurement (3 semester hours)
- Algebra and number theory (3 semester hours)
- Statistics and probability (3 semester hours)

Other courses carefully selected from the offerings of mathematics or statistics departments would include introductory statistics, calculus (with particular attention to the concepts underlying the standard topics of calculus), number theory, discrete mathematics, history of mathematics, and modeling (to provide and understanding of the ways in which mathematics and statistics are applied).

Professional development of middle grades teachers should focus on deepening mathematical knowledge in the domains of the CCSS. Until such time as all middle school teachers specialize in mathematics, the present K-6 certification of teachers makes it important that such professional development opportunities for teachers of grades 6-8 be extended to teachers of grade 5. As was the case for professional development of elementary school teachers, the professional development of middle grades teachers may take many forms.

- Teachers might form a professional learning community to focus deeply one topic for an extended period of time.
- Teachers of grades 6, 7, and 8 at one school might study how related topics develop across grade levels.
- A group of teachers might observe demonstration lessons taught by a mathematics specialist, discuss them, and plan additional lessons.
- Professional development can take place at a school location (during or after school hours) or on a college campus.

Regardless of the format of the professional development. it should include the discussion of student learning, common student misconceptions, ways that ideas in the CCSS are related, and the most useful representations of those ideas. The Content Standards of the CCSS should be viewed as inseparable from the Standards for Mathematical Practice. Teachers should learn to make use of technological tools -- whiteboards, tablets, virtual manipulatives, dynamic geometry software, graphing calculators, applets, websites, and multi-media materials -- and should critically evaluate the effectiveness and limitations of those tools in both their own learning and in supporting the learning of their students.

Chapter 6
High School Teachers

Chapters 6 provides recommendations to those engaged in the preparation or professional development of high school teachers (which MET II defines to be grades 9-12). The sequence of mathematics courses for pre-service teachers has two course-content versions: the short sequence (33 semester-hours) and the long sequence (42 semester hours). Both sequences have three components.

1. Essentials in mathematical preparation
2. Additional undergraduate (or professional development) mathematics courses
3. Essential mathematical experiences for practicing teachers

Although these courses might come from standard courses for mathematics (or other) majors, the chapter describes how such courses might be adjusted to better connect with high school mathematics. As a minimum, MET II recommends that the pre-service program include a three-course sequence in calculus, an introductory course in linear algebra, an introductory course in statistics, and 18 additional semester-hours of advanced mathematics, 9 of which are "explicitly focused on high school mathematics from an advanced standpoint."

Essentials of Mathematical Preparation

At present, high school mathematics often is presented as a collection of unrelated facts. Therefore, to support retention and connection of mathematical knowledge, instructors of courses that include prospective mathematics teachers should pay careful attention to building and guiding mathematical reasoning. Learners must be active participants in developing mathematics of those courses and must be required to reflect on their reasoning. The mathematics program of pre-service high school teachers should be drawn from three sources.

- Courses selected from those taken by a variety of undergraduate majors.
  These courses include: the standard three-semester introduction to single- and multi-variable calculus; introduction to linear algebra (since the CCSS includes operations on vectors and matrices in solving systems of equations); and statistics and probability (since the CCSS includes basic concepts in this area and the advanced/optional CCSS standards call for using probability and statistics in real-world decision making).
- Courses selected from those taken by mathematics majors.
  These courses include: introduction to proof (in which prospective teachers should analyze and construct proofs themselves); abstract algebra (in which teachers should gain an understanding of the how the properties of operations -- i.e., the field axioms -- determine the permissible manipulations of algebraic expressions; the real number system (in which teachers construct the real number system using the properties of operations; modeling (a CCSS emphasis); differential equations (a traditional course for engineers that is heavy on analytical techniques); group theory (one application of which is the CCSS approach to geometry though rigid motions); number theory (which includes modular arithmetics with their connection to cryptography); and history of mathematics (which reveals mathematics to be a living and evolving subject).
- Courses designed primary for prospective teachers: MET II calls for three courses that have a primary focus on high school mathematics (from an advanced standpoint). Chapter 6 suggests the content focus of four kinds of courses that would serve this end.
  - >> *Geometry and transformations*. This would address the CCSS approach to Euclidean geometry based upon translations, rotations, reflections and dilations. To increase teachers' understanding of the role played by the parallel postulate in Euclidean geometry, they should investigate geometries in which the parallel postulate does not hold.
  - >> *Analytic geometry*. One aspect of such a course would be to have teachers explore analytic proofs of standard high school geometry theorems.
  - >> *Complex numbers and trigonometry*. Such a course would use the complex number system to connect algebra and geometry and to view many trigonometric identities as algebraic identities in the complex numbers.
  - >> *Research experience*. The objective would be to have teachers develop mathematics both from the standpoint of mathematicians (by performing experiments, grappling with problems, building abstractions by reflecting upon experiments, and developing theories) and from the standpoint of teachers (by finding simple ways to make mathematics manageable by students, to see connections with school mathematics, and to build on students' ideas).

Important Additional Mathematics

Teachers will need to explore additional mathematics throughout their careers. Specifically, professional development courses should provide further investigation of curriculum-relevant mathematics.

- Teachers may need to further study statistics. This is especially true for teachers teaching Advanced Placement statistics or teaching the advanced statistics topics suggested in the CCSS.
- Many states are moving to require that students complete a fourth year of high school mathematics, a move that will require development of courses that are an alternative to pre-calculus, calculus, and statistics. To meet that need, teachers might benefit from the study of discrete mathematics and computer science in order to teach courses that build on the CCSS modeling and advanced topics standards.
- Teachers could benefit from the further study of the mathematics that they are teaching in the context of how it fits into the broader landscape of mathematics. A goal of that study would be "bringing mathematical coherence to high school mathematics, showing how a few general-purpose ideas and methods can be used across the entire high school spectrum of mathematics.'

Essential Experiences for Practicing Teachers

All teachers need to strengthen their mathematical knowledge for teaching. Many practicing teachers were prepared prior to the development of the CCSS. They will need opportunities to study CCSS-related content. In addition to traditional workshops and summer courses, three professional development formats are suggested.

1. Mathematics teachers' circles and study groups. Teachers' circles, in which teachers and mathematicians meet together regularly in informal sessions that address the isolation of teachers both from each other and from practicing mathematicians.
2. Immersion experiences. Usually held over a summer, these experiences provide teachers with a research experience in mathematics that helps them to understand the nature of doing mathematics. It also "reminds them that frustration, confusion, and struggle are all natural parts of being a learner."
3. Lesson study. In these studies, small groups of teachers and mathematicians, mathematics educators, and administrators collaboratively plan lessons. One or more members will then teach the lesson; other members observe. After debriefing and revision, the lesson can be taught to other students.

Caveat Emptor

This summary was prepared by Bob Kansky (robk@tribcsp.com), Professor Emeritus at the University of Wyoming. It does not critique the report's assumptions, methods, or conclusions. Also, although considerable effort was made to avoid distorting the intent of the report's authors, errors may have been introduced as a result of selecting and rewording of the report's text. Readers should consult the original document for further information.