Illustrating the Standards for Mathematical Practice: Congruence and Similarity through Transformations
FACILITATOR'S AGENDA

Session 1: Congruence and Similarity through Transformations 2 hours

Session Goals:

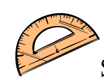
- To explore the Standards for Mathematical Content and Practice through video of classroom practice
- To consider how the Standards are likely to impact your mathematics program and to plan next steps
- To examine congruence and similarity defined through transformations
- To examine the use of precise language, viable arguments, appropriate tools, and geometric structure

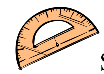
What is the point of this session?

To gain an understanding of the CCSS 8^{th} grade transformation based definition of congruence and similarity as well as the mathematical practices that deal with precise language, viable arguments, appropriate tools, and structure.

**The Notes section of each session agenda includes guidance and notes to support you in facilitating the activities. ✎ Indicates mathematics content information and support.*

Time	Materials	Activity	Notes
10 min	PPT slides 1-5 Standards for Mathematics Practice Handout	Introduction & Goals ○ Common Core State Standards – include standards for content & standards for practice (slides 1 & 2) ○ Goal for the Session (slide 3) ○ Considering the Standards for Mathematics Practice (slides 4-5)	Welcome the group and introduce the session, noting that it is situated in both the content and practice standards included in the CCSS (slide 2). Share the goals of the session (slide 3) and explain that the session will explore both types of standards through video of classroom practice. If the participants do not already know one another, also take this opportunity for everyone to introduce himself or herself. Distribute the Standards for Mathematics Practice Handout. Review the 2 slides (slides 4 & 5) on the Standards for Mathematics Practice, and ask if the participants have questions.

25 min	PPT slides 6-14 Static & Transformation-Based Conceptions Handout	**Defining Congruence & Similarity through Transformations** o Introduction to Congruence & Similarity through Transformations (slide 6) – Definition of Congruence & Similarity o Reflective writing: Your definitions of congruence & similarity (slide 7) o Definitions used in the standards (slide 8) – Static and Transformation-Based Conceptions o Static & transformation-based conceptions of similarity (slides 9-10) o Static & transformation-based conceptions handout (slide 11) o Categorize your definitions (slide 12) o Standards for Mathematics Content & Practice (slides 13-14)	<u>Definition of Congruence & Similarity (slides 7-8)</u> Have the participants individually reflect on and write about their definitions of congruence and similarity. Do not have them share their definitions at this point. They will share and classify their definitions after they discuss static & transformation-based conceptions (see slide 12). Then go over the definitions of congruence & similarity used in the standards. Note that the sequences of transformations pictured in slide 8 are not the only sequences possible to establish the congruence and similarity of figures. <u>Static & Transformation-based Conceptions of Similarity (slides 9-12)</u> There are various accurate conceptions of similarity. These conceptions can be categorized as either static or transformation-based. Static conceptions are likely to be more familiar, whereas transformation-based conceptions are closely aligned with those used in the standards. Note that Slide 9 has two parts. Click ONE TIME to show a comparison of the ratios of corresponding parts *between* two geometric figures. Click ONE MORE TIME to show the comparison between the ratio of two parts *within* a geometric figure and the corresponding ratio within another figure.  Slide 9 illustrates a traditional, static presentation of similarity, which is a comparison of two discrete figures. Our experiences with similarity as learners and as teachers have most likely been from a static/ discrete perspective—examining two or more figures as discrete and comparing corresponding side lengths and angle measures. This differs from the Common Core State Standards transformation-based definition which utilizes a dynamic, continuous perspective. The given triangles share two different relationships; namely, a "between-figure" relationship and a "within-figure" relationship. The "between-figure" relationship reveals that the pairs of corresponding sides are in the same proportion. The "within-figure" relationship reveals that the ratios of lengths within a figure are equal to ratios of corresponding lengths in a similar figure. The within figure relationship is sometimes referred to as the aspect ratio and the between figure ratio

			is the scale factor. Slide 10 contains an animation. Click ONCE to begin the animation that will run through the creation of several similar triangles.  Slide 10 illustrates a transformation-based perspective, in which the focus is on enlarging & reducing figures proportionally to create a class of similar figures. Slide 10 starts with the same 1/3 triangle that we just encountered in Slide 9, and when it passes through the similar 2/6 triangle from Slide 9 that triangle turns purple to match the one from Slide 9. The animation highlights the fact that this discrete triangle belongs to an infinite class of similar triangles, which include all of the ones for which the scale factor and side lengths are named in the animation, but also all of the triangles in between. In response to the question "What do you notice about the geometric structure of the triangles," teachers might notice: - All of the triangles are part of a class of similar triangles. - As the scale factor changes, the height and base change proportionally. - We can still examine the within and between relationships among the triangles. - The triangles are dilation images. - The triangles are bounded by the 2 given dilation lines (the hypotenuses & the bottom legs). - The angles remain constant (i.e., their measures are preserved) If teachers are not sure how to answer the question, ask them to list anything they notice or wonder about the set of triangles that the animation shows, and then point out the ways that they are attending to the geometric structure of those triangles. For example, the relationships between parts of the triangles, the relationships among the set of triangles, and specific geometric features of the triangles (such as the angles or dilation lines used to create them) are all parts of the geometric structure of the set of triangles. Slide 11 defines the terms "static" and "transformation-based," and provides a summary of these 2 conceptions of similarity. There is also a handout that matches this slide, which may be helpful to teachers as a reference: the Static & Transformation-Based Conceptions Handout.

			Note that the mathematical practice of "look for and make use of structure" emerges in this slide as geometric structure can be applied here, especially to the transformation-based conception. Slide 12 prompts teachers to share and categorize the definitions they wrote for congruence & similarity. Starting with congruence, have teachers share, classify, & provide rationale for each idea as static or transformation-based. Then do the same for similarity. Be sure to prompt for teachers' reasoning behind their classifications. For example, are their definitions focused on numeric or geometric relationships? Do their definitions attend to discrete figures or a class of similar figures? It is possible that some definitions may not contain enough information to readily classify. You might want to add a "not sure" category. It might be helpful to do this exercise on chart paper (using two charts—one for congruence and one for similarity) This allows for a public record of teachers' definitions, which we will return at the end of the session. It is not necessary to discuss accuracy or precision of the definitions at this point; that will come at the end of the session. <u>Standards for Mathematics Content & Practice (slides 13-14)</u> The 8th grade geometry content standards that define congruence and similarity from a transformation-based perspective frame the mathematical work in this session. Of particular relevance are the transformation-based definitions of congruence and similarity. Other content standards are also connected such as 7.G.1 "Solve problems involving scale drawings of geometric figures." Of the 8 standards for mathematical practice, the 4 that we will highlight in the remainder of this session are: constructing viable arguments & critiquing the reasoning of others, using appropriate tools strategically, attending to precision, and making use of structure.
20 min	PPT Slide 15-16 Rectangle problem handout Tracing paper,	Hannah's Rectangle Problem (Slide 15-16 and handout)	<u>Working on the Task</u> Have teachers work on Hannah's Rectangle Problem individually or in small groups, and then discuss it as a whole group using the focus questions on slide 16. Have tracing paper, rulers and scissors available. Try not to allot more than 10 minutes for teachers to solve the problem,

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<td>rulers, scissors</td>
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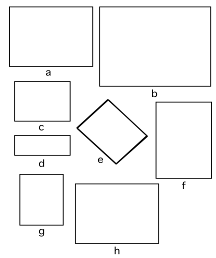

Task: Which rectangles are similar to rectangle a?

Hannah’s Rectangle Problem Discussion
- Which rectangles are similar to a? Construct a viable argument for why those rectangles are similar.
- Which definition of similarity guided your strategy, and how did it do so?
- What tools did you choose to use? How did they help you?
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so that there are at least 10 minutes for discussion. You can reassure teachers that they do not have to find ALL of the rectangles that are similar to rectangle a. As long as they explore a few, that will suffice. Some strategies (such as using tracing paper) are likely to help teachers progress through the problem faster than other strategies (such as measuring). However, it is more important for teachers to choose their own methods than to finish the problem.

<u>Discussing the Problem</u>
Which rectangles are similar to a?
- Rectangles b, c, e, g, and h are similar to a.

Viable arguments include the following:
- The rectangles that are similar have the same ratios. Teachers could point to ratios between rectangles (length rectangle 1: length rectangle 2 is equal to width rectangle 1: width rectangle 2). Or teachers could refer to ratios within rectangles (length: width rectangle 1 equals length:width rectangle 2).
- Rectangle h is congruent to rectangle a, therefore they are similar.
- Rectangles c, e, & g are all congruent (through a rotation).
- Rectangles b, c, e, g, & h can each be placed (through rotations and translations) so that they are dilations of rectangle a. In other words, dilated images of rectangle a can be created that are congruent to rectangles b, c, e, g, & h.
- For rectangle h, the congruent dilated image is dilated by a scale factor of 1 (because rectangles a and h are congruent). For rectangle b the congruent dilated image is dilated by a scale factor of 4/3. For rectangles c, e, and g the congruent dilated image is dilated by a scale factor of 3/2.
Possible facilitator probes around viable arguments:
- Are you making a conjecture or do you have evidence?
- Does your line of argument work for any set of rectangles?

Definitions that guided teachers’ strategies:
- Refer back to the list generated from slide 12.
Possible facilitator probes for definition question:
- Did you use a definition not listed on our chart?
- Are there connections between the strategies people used to solve the problem and whether their definitions were more
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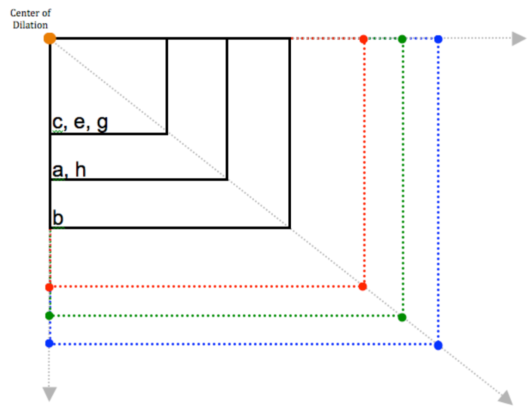
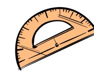
			"static" or "transformation-based?" - How does a change in definition from static to transformation-based (or visa versa) change the line of reasoning for your viable argument? Tool use: - Both tracing paper and rulers can be used as measurement tools for comparing within-figure or between-figure ratios when focusing on a static definition of similarity. - Tracing paper can be used as a measurement tool to super-impose one rectangle on top of another, allowing for comparison of how two figures relates to each other. In this way, tracing paper allows the rotation, reflections, or translations of rectangle A to the location of the other rectangles. - Both tracing paper & rulers can be used to determine whether a dilation relates the two figures. Most teachers are unlikely to use the tools in this way, but this method will be demonstrated by a student (Randy) in the upcoming video clip. Possible facilitator probes for tool use: - Is it possible to solve the problem without a ruler? - How could using tracing paper help you reason through the problem?
40 min	PPT Slide 17-21 Lesson graph Transcript Video clip Tracing Paper	Preparing for the Video Clip (slides 17-19) o Intro to Watching Video & Norms o Intro to Lesson Graphs o Intro to the Video Clip Watch & Discuss the Video Clip Unpacking Randy's method (slide 20) • What did Randy do? (What was his method?) • Why might we argue that Randy's conception of similarity is more transformation-based than static? • What mathematical practices does he employ? - What mathematical argument is he using? - What tools does he use? How does he use them strategically? - How precise is he in communicating his reasoning?	<u>Norms for Watching Video (slide 17):</u> It is important to briefly discuss the purpose of watching and analyzing video cases and to set norms for watching video. Throughout the discussions that follow, facilitators might need to remind participants of these norms, if the comments become judgmental. <u>Intro to Lesson Graphs (slide 18):</u> Distribute Hannah's Lesson Graph and explain that a lesson graph is a 1-page overviews of a lesson. The portion of the page devoted to each lesson activity is (roughly) proportional to the percentage of time that lesson activity took within the lesson. The grey segments represent independent/small group work by students, and the white segments represent whole class work. The yellow highlighted section indicates what is happening in the clip that will be viewed during this session. Give teachers time to look at the lesson graph *prior* to watching the clip, to get a sense of the entire lesson and what came before and after the selected segment. After viewing the clip, the lesson graph can serve as a reference and/or

		Representing an Infinite Set of Similar Rectangles (slide 21) ○ Summarize – Dilation creates an infinite set of similar rectangles (Slide 21)  	evidence during teachers' analysis of what happened in the clip. <u>Intro to the Video Clip</u> (slide 19) Note that this clip comes from an 8th grade lesson, filmed in the fall. The students are working on the same problem we just worked on. Distribute the transcript. Explain that the video clip is subtitled, but a transcript is provided because it can be quite useful during the discussion of the clip. Encourage teachers to turn to the transcript for evidence of what really did (and did not) happen. <u>Watch & Discuss the Video Clip</u> (slide 20). Teachers should watch clip and then discuss the questions on slide 20. There are many possible structures to use for discussing the video clip in this session. For example, teachers could think/write about the clip individually, share with a partner or small group, then discuss as a whole group (think, pair, share). You might ask them to consider/discuss one guiding question at a time, or all of the questions, or divide the questions among groups of teachers. These decisions will depend on time and other factors. This session is paced assuming that there will likely only be time to watch the clip once. <u>Re-Creating Randy's Method</u> Have participants use tracing paper to re-create Randy's method and explain his reasoning. Randy's approach is an example of a transformation-based perspective, in which he is using dilation to attend to the dynamic relationship between the original figure and a family of similar rectangles. Randy constructs dilation lines to determine whether the original figure can be extended in such a way that the corresponding sides and diagonals fall along those dilation lines. Rather than choosing to measure the sides of the rectangles and compare static figures, Randy's approach does not involve any sort of measurement and rather relies on a visual inspection of accurate scaling. Randy uses several mathematical practices as he solves this problem: • He recognizes the **geometric structure** and uses center of dilation and dilation lines to determine similarity • He uses patty paper and dilation as **tools** to determine mathematical similarity • He uses **precise mathematical language** at times (center of

			dilation) and imprecise mathematical language at other times as he works to explain his method • He creates a **viable argument** and justifies his conclusions for determining similarity using dilation and explains how to determine if a rectangle is not similar (a counter-example).  <u>Representing an Infinite Set of Similar Rectangles (slide 21)</u> After unpacking Randy's method, use the representation on Slide 21 to summarize by noting that his method provides a window into the idea that an infinite set of similar figures results from dilation. The representation on slide 21 is one way to illustrate this idea. The slide highlights the geometric structure of similar rectangles. Notice that we have organized the rectangles so that they are "nested." They share a common center of dilation, as indicated in the slide. The slide depicts 3 of the lines of dilation that they share. These 3 lines extend from the center of dilation and go through the rectangles' corresponding vertices (along the length, width and diagonal). The blue rectangle that dilates (by growing and shrinking) as the slide's animation plays demonstrates that there is an infinite set of similar rectangles along the lines of dilation. Rectangles c, e and g are congruent rectangles, which can be demonstrated by rotating the figures shown on the Rectangle Problem handout. The same is true for rectangles a and h. As can be seen in slide 21, similar rectangles can be formed along the lines of dilation. This set of similar rectangles (a, b, c,e,g,and h) can be understood as dilation images or each other.
15 min	PPT slides 22-23 Definitions handout	Summary: Reconsidering Definitions of Similarity Complete the Definitions Handout (slide 22) - Consider the possible definitions for similarity that are listed. Add any definitions you'd like, either from your curriculum or personal definitions. - What language appears to be imprecise? Precise? - What possible student misinterpretations can you imagine? Try to draw or represent them.	Distribute the Definitions Handout. Teachers can work individually, in pairs or small groups to fill out the chart prior to a whole group discussion. The focus of this activity is to consider the precision of language in definitions and how they can be interpreted or misinterpreted by students. Some common definitions for similarity as well as the CCSS definition are included in the handout (illustrated in slide 22). A good definition is precise enough to be useful in solving problems—"Same shape, different size" is not very useful.

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<td>Considering Features of Similarity Definitions
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<tr><th>Possible Similarity Definitions</th><th>Imprecise Language</th><th>Precise Language</th><th>Possible Student Misinterpretations</th></tr>
<tr><td>Two figures are similar if they are the same shape and different sizes.</td><td></td><td></td><td></td></tr>
<tr><td>Two figures are similar if they have the same shape but not necessarily the same size</td><td></td><td></td><td></td></tr>
<tr><td>Two figures are similar if they have corresponding angles equal and corresponding line segments proportional.</td><td></td><td></td><td></td></tr>
<tr><td>Two figures are similar if one is the same as an enlargement or reduction of the other.</td><td></td><td></td><td></td></tr>
<tr><td>Two figures are similar if the second can be obtained from the first (congruent) by a sequence of rotations, reflections, translations and dilations.</td><td></td><td></td><td></td></tr>
<tr><td>(curriculum definition)</td><td></td><td></td><td></td></tr>
<tr><td>(personal definition)</td><td></td><td></td><td></td></tr>
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○ Field Guide to a transformation-based definition of congruence and similarity (slide 23)</td>
<td>You might also ask teachers to include either the definition of similarity from their own curriculum or their own personal definition. It may be helpful to refer back to the definitions they generated earlier in the session, to see how they compare to the ones that are listed (see slide 12).

In addition to the handout for teachers, there is a “key” for facilitators at the end of this agenda (page 11 and 12).

<u>Field Guide: A Resource for Your Practice</u>
Distribute the Field Guide to a Transformation-Based Definition of Congruence and Similarity. This guide is intended for teachers to keep as a resource for their own classroom practice, which they may also share with their students. Ideally the field guides would be laminated. Have the teachers examine and briefly discuss the field guide.

Possible discussion prompts for the field guide:
- What in this field guide represents pieces of what Randy did in the video?
- Why are the non-examples non-examples?
- What would other non-examples be?
- What is the relationship between the definition of congruence and the definition of similarity?
- How might you use this field guide?</td>
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<td>PPT slides 24-28</td>
<td>End of Day Reflections & Acknowledgements
○ End of Day Reflections (slide 24)
○ Acknowledgements

- Noyce Foundation
- WestEd
- NCSM Series contributors</td>
<td>End of Day Reflections
Ask teachers to reflect on the 2 questions listed on slide 24. In particular, teachers should think about the implications of this session on their instructional practice and on supporting their students in learning this content.

Acknowledgements (slides 25-27)
Please briefly acknowledge the Noyce Foundation. This online PD series was made possible by a generous grant from the Noyce Foundation to NCSM.

In addition, video for this module comes from the project Learning and Teaching Geometry, funded by the National Science Foundation and soon to be published by WestEd. WestEd has graciously granted</td>
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			NCSM permission to utilize the video clip and supporting resources for this module. Contributors to the online PD series (of which this session is a part) are listed on slide 27.

Possible Similarity Definitions	Imprecise Language	Precise Language	Possible Student Misinterpretations
Two figures are similar if they are the same shape and different sizes.	"same shape and different sizes"	n/a	 Both are triangles (same shape). A right triangle and an equilateral triangle are both the "same shape" (triangles) and could be different sizes (as measured by area, perimeter, or how they look) Two congruent triangles are not similar
Two figures are similar if they have the same shape but not necessarily the same size	"same shape but not necessarily the same size"	"not necessarily" is slightly more precise than definition #1 with respect to the size relationship between similar figures; e.g., congruent figures are similar.	 A right triangle and an equilateral triangle are both the "same shape" (triangles) and could be different sizes (as measured by area, perimeter, or just by how they look)
Two figures are similar if they have corresponding angles equal and corresponding line segments proportional.	"line segments"	"corresponding angles equal and corresponding line segments proportional"	Only the sides (where line segments can be seen) on figures need to be in proportion, not other distances within the figure.
Two figures are similar if one is the same as an enlargement or reduction of the other.	"same" "enlargement or reduction"		 All sides enlarged but not by the same factor Only the sides (where line segments can be seen) on figures need to be in proportion, not other distances within the figure.

Two figures are similar if the second can be obtained from the first (congruent) by a sequence of rotations, reflections, translations and dilations.		"if the second can be obtained from the first (congruent) by a sequence of rotations, reflections, translations and dilations.	

COMMON CORE STATE STANDARDS FOR MATHEMATICS

The *Standards for Mathematical Practice* is a document in the CCSS that describes different types of expertise students should possess and mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education.

The first of these are the National Council of Teachers of Mathematics, NCTM, process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up:* adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately) and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy). This paper combines information from those sources and lists what students will be doing when they demonstrate mathematical proficiency.

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.
Mathematically proficient students:

- start by explaining to themselves the meaning of a problem and looking for entry points to its solution.
- analyze givens, constraints, relationships, and goals.
- make conjectures about the form and meaning of the solution.
- plan a solution pathway rather than simply jumping into a solution attempt.
- consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution.
- monitor and evaluate their progress and change course if necessary.
- who are older might, depending on the context of the problem:
  - ✓ transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need.
  - ✓ explain correspondences between equations, verbal descriptions, tables, and graphs.
  - ✓ draw diagrams of important features and relationships, graph data, and search for regularity or trends.
- who are younger might:
  - ✓ rely on using concrete objects or pictures to help conceptualize and solve a problem.
  - ✓ check their answers to problems using a different method.
- continually ask themselves, "Does this make sense?"
- understand the approaches of others to solving complex problems and identify correspondences between approaches.

2 Reason abstractly and quantitatively.
Mathematically proficient students:

- make sense of quantities and their relationships in problem situations.
- bring two complementary abilities to bear on problems involving quantitative relationships:
  - ✓ *decontextualize* - to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents and
  - ✓ *contextualize* - to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.

- use quantitative reasoning that entails habits of creating a coherent representation of the problem at hand:
  - ✓ considering the units involved,
  - ✓ attending to the meaning of quantities (not just how to compute them), and
  - ✓ knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students:

- understand and use stated assumptions, definitions, and previously established results in constructing arguments.
- make conjectures and build a logical progression of statements to explore the truth of their conjectures.
- analyze situations by breaking them into cases.
- recognize and use counterexamples.
- justify their conclusions, communicate them to others, and respond to the arguments of others.
- reason inductively about data, making plausible arguments that take into account the context from which the data arose.
- compare the effectiveness of plausible arguments.
- distinguish correct logic or reasoning from that which is flawed and, if there is a flaw in an argument, explain what it is.
  - ✓ Elementary students construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.
  - ✓ Later, students learn to determine domains to which an argument applies.
- listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve arguments.

4 Model with mathematics.

Mathematically proficient students:

- apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.
  - ✓ In early grades, this might be as simple as writing an addition equation to describe a situation.
  - ✓ In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community.
  - ✓ By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another.
- apply what they know to make assumptions and approximations to simplify a complicated situation, realizing that these may need revision later.
- identify important quantities in a practical situation.
- map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.
- analyze those relationships mathematically to draw conclusions.
- routinely interpret their mathematical results in the context of the situation.
- reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students.

- consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, protractor, calculator, spreadsheet, computer algebra system, a statistical package, or dynamic geometry software.
- are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations.
- who are in high school:
  - ✓ analyze graphs of functions and solutions generated using a graphing calculator.
  - ✓ detect possible errors by strategically using estimations and other mathematical knowledge.

- ✓ when making mathematical models, know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data.
- identify relevant external mathematical resources (e.g., digital website content) and use them to pose or solve problems.
- use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students:

- try to communicate precisely to others.
- try to use clear definitions in discussion with others and in their own reasoning.
- state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.
- carefully specify units of measure and label axes to clarify the correspondence with quantities in a problem.
- calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.
  - ✓ In the elementary grades, students give carefully formulated explanations to each other.
  - ✓ By high school, students have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students:

- look closely to discern a pattern or structure.
  - ✓ Young students might notice that three and seven more is the same amount as seven and three more or they may sort a collection of shapes according to how many sides the shapes have.
  - ✓ Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for the distributive property.
  - ✓ Older students, in the expression $x^2 + 9x + 14$, can see the 14 as 2 x 7 and the 9 as 2 + 7.
  - ✓ They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems.
- step back for an overview and shift perspective.
- see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. (e.g., They see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.)

8 Look for and express regularity in repeated reasoning.

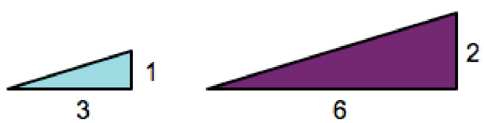
Mathematically proficient students:

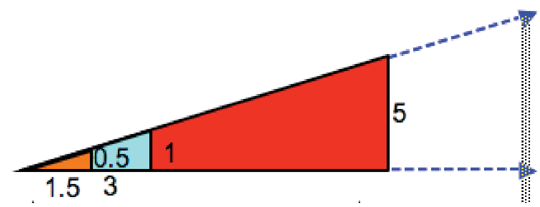
- notice if calculations are repeated.
- look both for general methods and for shortcuts.
  - ✓ Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeated decimal.
  - ✓ By paying attention to the calculation of slope as they repeatedly check whether the points are on the line through (1,2) with a slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$.
  - ✓ Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead high school students to the general formula for the sum of a geometric series.
- maintain oversight of the process of solving a problem while attending to the details.
- continually evaluate the reasonableness of their intermediate results.

Static and Transformation-Based Conceptions of Similarity

STATIC:
Similarity is conceptualized in discrete terms as a numeric relationship between 2 figures.

TRANSFORMATION-BASED:
Similarity is conceptualized as enlarging or reducing figures proportionally to create a class of similar figures.





Evidence of Static Conceptions

- There is a focus on comparison of numerical relationships between corresponding parts of similar figures
- Attention is on numerical relationships between corresponding measures in similar figures (side lengths and angle measures).
- Attention is on ratios of lengths within a figure & noticing that ratio remains constant for other figures

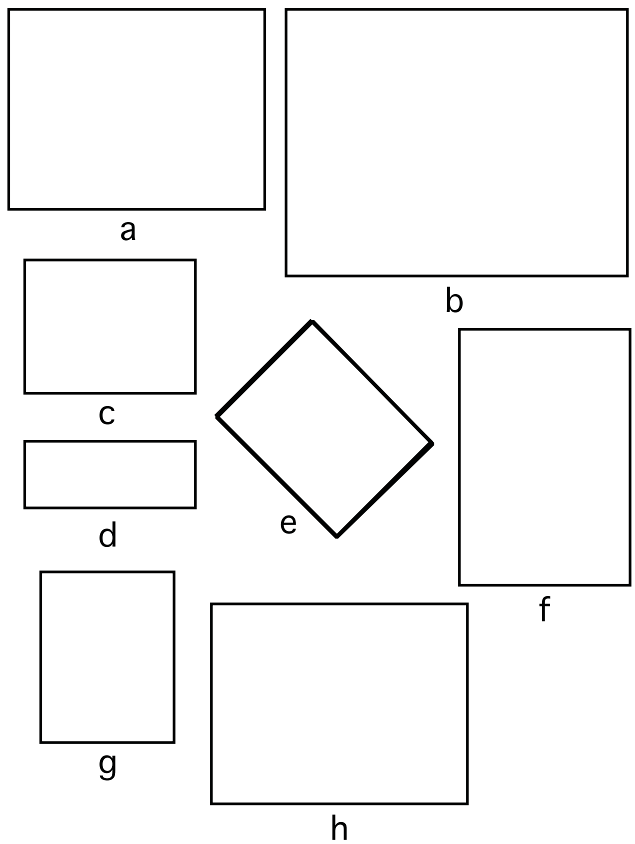
Evidence of Transformation-based Conceptions

- There is a focus on geometric transformations that result in similar figures
- Attention is on geometric relationships among similar figures, including:
    - Dilating (stretching/shrinking) to create or compare figures
    - Translating, rotating & reflecting to create or compare figures
- Attention is on all possible figures in a similarity class enabled by visual representations of dilating figures

ADD YOUR OWN EXAMPLES, NOTES ABOUT STATIC CONCEPTIONS	ADD YOUR OWN EXAMPLES, NOTES ABOUT TRANSFORMATION-BASED CONCEPTIONS

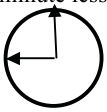
Rectangle Problem

Which rectangles are similar to rectangle a? Explain the method you used to decide.



[45 minute lesson]

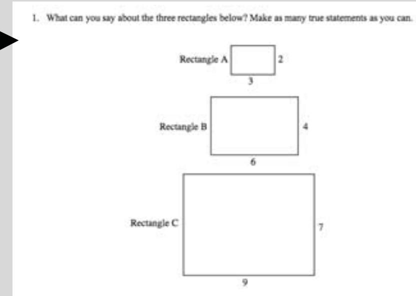
Session 3 Lesson Graph: Similarity Problems, Hannah Slovin, Class 1 (8th Grade)



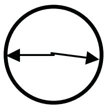
8 minutes

Warm-up ***Students work on Similarity I #1 individually***

Teacher gives a warm-up task
Teacher passes out tracing paper and says, "Look for the obvious. Look for the more interesting."



Students work individually & teacher walks around the room.



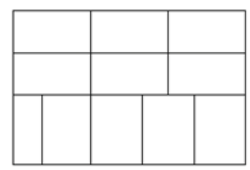
17 minutes

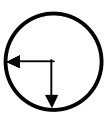
Whole Group *Students present their ideas & the teacher writes them on the Document Camera.*

Kiana: The width of the three squares is going by 3's.
Teacher rephrases using "rectangles," Points out 3, 6, 9.
Heidi: The opposite lines on each of the rectangles are parallel.
Pono: Area of A is 6 square units.
Meagan: Rectangle B is 4 times bigger than Rectangle A. You can fit 4 of A into B.
T asks Meagan to show this, noting some students thought the area was twice as big. Meagan uses tracing paper to fit A into B 4 times.
T: Why might somebody think the area is twice as big?
Nicollete: Because 4 is 2x2 and 6 is 3x2.
Meagan: The perimeter is twice as big.

Teacher asks Blaine to share what he wrote down.
Blaine: They all have the same proportions.
James: They're all to scale with each other. Like A is ¼ the size of B.
T: So you're checking the area.
Jenson: If you increase the rectangle by a percentage, it'd be the same figure. Just bigger or smaller
Jenson shows his idea on the Document Camera: If you dilate or make it bigger, it'd be the same rectangle. Jenson argues all 3 rectangles are similar. Some students agree, others disagree.

Olivia says you can fit a combination of As & Bs inside of C, if you rotate them. She shows her idea on the Document Camera. The teacher notes that in a dilation, shapes maintain their orientation. Olivia responds, "The shapes fit inside but it's not a dilation. T asks for the area of C. *63.* So how many times does A fit into C? *10 ½ times*.
Ethan. C is not a dilation of B because the ratio of width to length is not the same. B is a dilation of A because if you multiply the length of A by 2, you get the length of B. And if you multiply the width of A by 2, you get the width of B. For B & C, the ratio of 4:7 is 1:1.75 and 6:9 is 1:1.33.
Student: If C were 6 instead of 7, it would work. 3,6,9 is a pattern. But 2,4,7 messes it up.



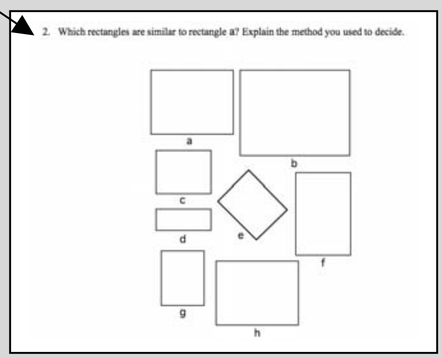


13 minutes

Small Groups *Students work on Similarity I # 2 in small groups*

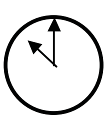
Teacher passes out problem #2 and tracing paper.

T: Who remembers anything about similar figures?
Randy: Same shape but not same size.
T: But A & C from Problem 1 are the same shape.
Randy: They're not similar because they don't continue the pattern.



Olivia says her group found a rectangle congruent to A, but they don't know if it's similar. T says this as an important question to ask the class.

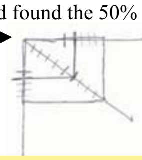
Randy talks to the T about finding similar rectangles using dilation.



7 minutes

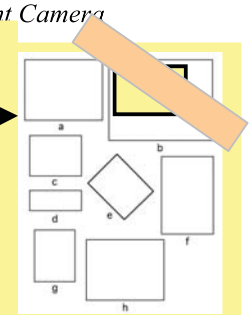
Whole Class *Students present their ideas & the teacher writes them on the Document Camera*

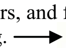
Payne says C is similar b/c it's a 60% dilation of A. Shows how he lined up the upper left corners, and found the 50% point by folding.



Randy shows how he compared A & B. He traced A, lined up the centers of dilation (upper left corners), and used a straight edge to see if the points were on the same line.
Teacher asks Randy to demonstrate for a rectangle that's not similar to A. Randy shows that for D, when you draw the dilation line, the points are "off."

T asks how Randy's method could be used on the triangle problem. Andrew says you can put the original triangle on top of the other ones and measure them. T notes many triangles are turned.





Clips 1 and 3

A student says H is the same as A. This leads to a conversation about whether congruent shapes are similar. The ratio is 1:1. Teacher concludes: "It's a special kind of similarity. It would be considered similar."

Teacher assigns homework: Complete the triangle problem and other problems.

Randy Transcript

Randy: So what I did was I traced…

Teacher: So you can start with a clean paper if you want, Randy. That will help.

Randy: So I traced A like this. I traced A. And then, I went around with a straight edge. And I went on it like that. And I lined up the center of dilation. Which, I chose the same thing as Payne. Yeah. And then I lined it up. And I looked with a straight edge to see if this point and this point is the same. And then this one and this one. And then if this one and this one was the same. And then, if it was the right one, I traced it. Yeah. Okay. And then I went to the different ones.

Teacher: Randy, can you show us one that doesn't work?

Randy: D. D doesn't work.

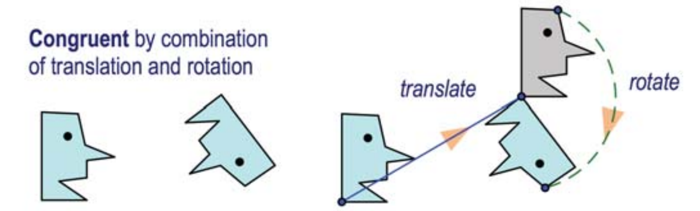
Teacher: Where you decided it was not similar.

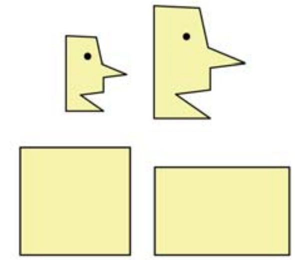
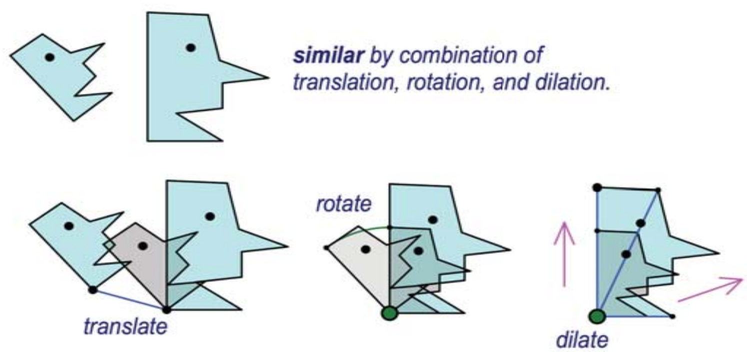
Randy: Because if you drew the dilation line, like that, and then the ones that were similar, they're off by like that much.

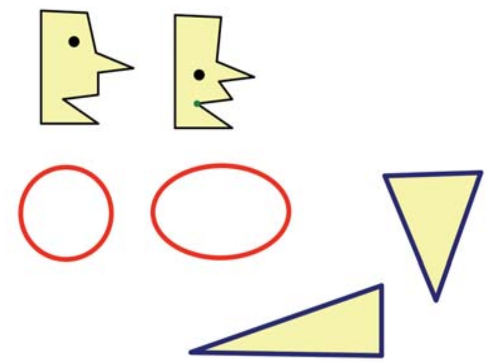
Teacher: So does anyone have questions for Randy about that method? How would you use that method to figure out with the triangle?

Possible Similarity Definitions	Imprecise Language	Precise Language	Possible Student Misinterpretations
Two figures are similar if they are the same shape and different sizes.			
Two figures are similar if they have the same shape but not necessarily the same size			
Two figures are similar if they have corresponding angles equal and corresponding line segments proportional.			
Two figures are similar if one is the same as an enlargement or reduction of the other.			
Two figures are similar if the second can be obtained from the first (congruent) by a sequence of rotations, reflections, translations and dilations.			
(Curriculum definition)			
(Personal definition)			

Field Guide to a Transformation-based definition of congruence & similarity

<table>
<tr><th></th><th>Imprecise Language</th><th>Definition</th><th>Properties</th><th>Examples</th><th>Non-examples</th></tr>
<tr><td>CONGRUENCE</td><td>The same, equal, "same shape and same size"</td><td>A two- dimensional figure is congruent to another if the 2nd can be obtained from the 1st by a combination of translations, rotations, and reflections.</td><td>For all points A, B and angles C
$|AB| = |A'B'|$
$m\angle C = m\angle C'$</td><td>

Note: many possible sequences of transformations could show this congruence</td><td>
(these rectangles have the same area, but are not congruent)</td></tr>
<tr><td>SIMILARITY</td><td>Stretch, scaled, resized, shrink, expand, "same shape"</td><td>A two- dimensional figure is congruent to another if the 2nd can be obtained from the 1st by a combination of congruence and dilation.</td><td>For all points A, B and angles C
$|AB| = k|A'B'|$
$m\angle C = m\angle C'$</td><td>

Note: many possible sequences of transformations could show this similarity</td><td></td></tr>
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