NATIONAL COUNCIL OF SUPERVISORS OF MATHEMATICS

NCSM JOURNAL of Mathematics Education Leadership

VOLUME 10, NUMBER 1 SPRING 2007

Table of Contents

State Mathematics Curriculum Standards and Reasoning

Ok-Kyeong Kim, Western Michigan University
Lisa Kasmer, Portage Public Schools

Since the release of the National Council of Teachers of Mathematics document, *Curriculum and Evaluation Standards for School Mathematics,* in 1989, states began to develop their own standards to set expectations for their students. Recently, many states updated their standards to incorporate new demands and trends in mathematics education (see Reys, 2006). In an effort to assess reasoning expectations in the state mathematics curriculum standards (state standards hereafter), we reviewed 35 state standards from kindergarten to eighth grade (Authors, 2006). In doing so, we focused on the extent and nature of emphasis on reasoning in five content areas (i.e., number and operations, algebra, geometry, measurement, and data analysis and probability), how grade level expectations (GLEs) related to reasoning are organized in the state standards, and overall characteristics of emphasis on reasoning across state standards. In this paper, based on the results of the review we discuss expectations we can have from the state standards in terms of reasoning as well as issues to consider in order to better promote reasoning. Let us begin our discussion by elaborating what reasoning is and what we mean by reasoning in this paper.

Reasoning and Its Importance

There seems to be a wide agreement on the importance of reasoning in mathematics teaching and learning. Reasoning is a process standard emphasized throughout the NCTM documents (NCTM, 1989, 2000). Ball and Bass (2003) state that mathematical learning cannot be considered without reasoning. To reason mathematically is fundamental to learning mathematics with understanding. When reasoning is effectively promoted through justifying results, developing ideas, predicting results, or making sense of observed phenomena, students can develop a deeper understanding of mathematical ideas. In turn, this deeper understanding equips students to enhance their mathematical reasoning. This way of learning mathematics will result in better learning outcomes. In this sense, the NCTM argues, “Reasoning and proof should be a consistent part of students’ mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts” (2000, p. 56).

While people agree that reasoning is important in the teaching and learning of mathematics, as Duval (1998) argues, there seems to be a wide range of ideas on what reasoning means. Reasoning is a broad and general term. According to Duval, “[A]ny process which enables us to draw new information from given information is considered as reasoning” (p. 45). Because of this broadness of reasoning, researchers, curriculum developers and teachers interpret reasoning diversely. For example, *Principles and Standards for School Mathematics* (NCTM, 2000), which includes reasoning and proof as one of the five process standards, emphasizes the importance of: making and investigating mathematical conjectures; developing and evaluating mathematical arguments; and selecting and using various types of reasoning and methods of proof. The *Trends in Mathematics and Science Study* (TIMSS)’s assessment framework on reasoning includes the following elements: 1) hypothesize/conjecture/predict, 2) analyze, 3) evaluate, 4) generalize, 5) connect, 6) synthesize/integrate, 7) solve non-routine problems, and 8) justify/prove (Mullis, Martin, Smith et al., 2001). When analyzing mathematics curricula in terms of reasoning, Stylianides and Silver (2004) focus on the process of proving, that is:

TABLE 1: ***Number of state curriculum documents that include GLEs in content strands in each component of reasoning by grade***

	K	1	2	3	4	5	6	7	8	K-8
Prediction	8	17	24	24	24	26	22	25	27	35
Generalization	2	1	8	5	9	10	10	12	12	21
Verification	2	1	2	5	7	7	6	6	13	21
Justification	1	1	8	12	14	23	20	19	24	31
Conclusion/Inference	1	6	9	12	13	16	15	16	17	26
Making Conjecture	0	0	1	2	5	7	6	13	10	19
Testing	1	1	4	6	12	10	6	9	7	20
Making Argument	0	0	1	0	2	6	3	7	11	14
Evaluation	0	0	3	2	2	7	9	9	14	20

identifying a pattern, making a conjecture, providing a proof, and providing a non-proof argument. Ball and Bass (2003) view reasoning as a process of inquiry and a process of justification. The former is used for "discovering and exploring new ideas" and the latter is used for "justifying and proving mathematical claims" (p. 30). Duval considers reasoning for extension of knowledge, for proof, and for explanation in his theorization of teaching and learning of geometry.

We also find various approaches to reasoning surfaced in state standards: *reasoning as meaning making, reasoning used in problem solving,* and *reasoning for verification*. First, reasoning is required for making meaning, concept development, connections among concepts, and relationship building. This broad approach to reasoning seems similar to what Duval refers to reasoning for "extension of knowledge" (p. 38). Second, reasoning is used in various phases of problem solving including: 1) analyzing problem situations, 2) developing and applying strategies, 3) selecting and applying strategies and mathematical ideas, 4) explaining strategies, and 5) checking the reasonableness of the results in the problem context. This approach to reasoning could be part of what Ball and Bass refer to reasoning for inquiry. Finally, reasoning can be considered as a thought process through which students make and test conjectures, prove or disprove them, and draw conclusions. This also includes prediction, argumentation, test, justification, verification, validation, evaluation, and generalization. In this paper, reasoning pertains to mainly *reasoning for verification* as this is a more common interpretation of reasoning and more specific than the other two approaches in state standards.

Expectations of Reasoning in State Standards

Overall, it is evident that state standards acknowledge the importance of reasoning. Many state standards documents either include a reasoning standard to address reasoning expectations besides those in content strands, or explicitly state that reasoning should be incorporated throughout content strands. State standards also provide various reasoning expectations, in some cases with specific examples. While overall efforts to incorporate the significance of reasoning in state standards are observable, many state standards fail to address reasoning in a thorough and comprehensive manner. Based on our findings, here we discuss expectations we can have from reasoning expectations in state standards in order to help better promote reasoning in the classroom.

First of all, it is important that state standards explicitly address what they mean by reasoning, what aspects of reasoning are expected and why, and how such reasoning expectations could be accomplished. Our findings show that state standards rarely document this even though the importance of reasoning is addressed and that such clarification is left to readers, which causes vagueness and inconsistency of reasoning expectations. A clear notion of reasoning and a solid plan for specific expectations of reasoning are required before listing reasoning expectations in each grade and in each content strand. It will also help develop state standards in ways suggested below.

Second, state standards should address reasoning in a coherent, consistent, and connected approach. We find that reasoning expectations in many states are addressed in a fragmented manner, rather than systemically and holistically. In fact, reasoning expectations are provided in the state standards with a great variation in terms of grade level, content strand, and state. For example, primary grades have a minimal number of reasoning expectations overall (see Table 1 and Table 3). Number and operations and measurement strands include a considerably fewer number of reasoning expectations than other content strands while the strand of data analysis and probability has an extensive number of reasoning expectations (see Table 2). Grade levels in which each state document addresses reasoning expectations in content strands also vary across states (see Table 3). There does not appear to be a cohesive plan in the K-8 state standards to promote reasoning.

Such inconsistency is also noticeable when comparing various reasoning expectations across states. Expectations pertaining to prediction (e.g., "predict the results of putting together or taking apart two-dimensional and three-dimensional shapes") are the most prevalent among the reasoning expectations, followed by expectations pertaining to justification, while making arguments, proving or disproving, and using counterexamples to refute claims have less attention in the state standards.

There is also a discrepancy between components of state standards documents when addressing reasoning aspects. For example, some states have sections delineated as 'benchmarks' and 'performance indicators' to address grade level expectations. Some of the benchmark statements do not specify reasoning aspects, but their corresponding performance indicators support reasoning. This discrepancy is also found when comparing GLEs and their examples. There are cases that a GLE has reasoning aspects but the example promotes mainly procedure, or a GLE does not specify any reasoning aspects but its example requires reasoning approaches (for detail see Authors, 2006).

In order to support reasoning in an effective way, state standards should address reasoning with a deliberate plan. To list a few, reasoning should be addressed not only in process standards, but also in content strands; reasoning GLEs and their sub-GLEs should be coherent; and examples should be aligned with reasoning expectations when they are used. In addition, reasoning GLEs in the state curriculum standards should have consistency across grades and content areas. Our findings show that even in states with explicit reasoning GLEs, a particular GLE does not appear across grades and content strands. For example, an important expectation such as 'develop arguments' is provided only in one or two grades in one content strand in most states. To promote reasoning in all grades and throughout various content areas, it is suggested that essential reasoning GLEs be provided in a consistent manner.

Connections among reasoning GLEs in the state standards should also be considered. An isolated reasoning aspect alone is not sufficient to promote a deeper level of reasoning. Reasoning GLEs should be presented along with other reasoning GLEs that are related to them. In fact, there are state curriculum standards that provide 'develop arguments' without 'evaluate arguments' or 'justify arguments' and vice versa. In other states, while making predictions appears often, testing, evaluating, or justifying predictions are very rare. In order to provide systemic reasoning GLEs, multiple aspects of reasoning, as those discussed in this paper, should be considered and these aspects should be addressed in relation to others.

TABLE 2: ***Number of state curriculum documents that include GLEs in each component of reasoning by content strand***

	Number	Algebra	Geometry	Measurement	Data/Prob
Prediction	10	19	21	4	32
Generalization	1	21	3	0	2
Verification	12	7	12	2	5
Justification	15	14	13	8	20
Conclusion/Inference	0	7	3	0	26
Making Conjecture	2	0	10	0	17
Testing	4	0	10	2	15
Making Argument	0	0	6	1	10
Evaluation	2	1	2	2	17

Third, reasoning GLEs need to be clear and specific. The Council of Basic Education describes specificity as "language that describes what is the most essential for students to learn using sufficient detail to convey what is expected without dictating instructional strategies" as well as "an aspect of rigor" (Joftus & Berman, 1998, p. 19). The American Federation of Teachers (2003) also suggests that state standards "must be clear and specific enough" for related personnel to understand and to lead a core curriculum.

TABLE 3: *Grade levels in which each state curriculum document addresses components of reasoning in content strands*

	Prediction	Generalization	Verification	Justification	Conclusion /inference	Making Conjecture	Testing	Making Argument	Evaluation
AL	5, 8	-	8	2, 8	-	-	-	8	-
AK	3-5, 7-8	-	-	3-8	4-8	-	-	-	4-8
AZ	1-8	-	4-8	-	-	-	-	-	2-3
AR	1-8	6-8	5-8	3-8	1-8	-	4	5-8	5-6
CO^	3-8	-	-	3-8	3-8	3-8	3-8	5-8	5-8
DD*	K-7	2, 4, 6-8	3-4, 8	2-4, 6-8	2, 6	7	2, 7	8	8
DC*	K-5, 7-8	5-8	1-5, 8	-	1-2, 4, 6-7	7	4, 7	-	7
FL	K-8	K, 2-5, 7	K, 8	4-5, 8	7-8	6-8	4-5, 8	-	8
GA	6	-	2-3	5-6	7	-	-	-	-
HI	2-5	2-3, 7-8	5	2-8	2-5, 7-8	4-5	2-5	4-5	5-8
ID	K-8	-	-	3-8	-	4-5	3-4	-	-
IN	1-2, 6-8	-	5-6, 8	5-8	-	-	7	-	8
KS	1-8	K-4, 6-8	-	6-7	2-8	-	-	5-8	8
LA	1, 4-6, 8	8	-	8	-	-	4-6	-	-
MI	1-2	-	8	5-8	-	-	-	-	-
MN	2-3, 6, 8	-	-	-	5-6	-	-	-	-
MS	1, 3, 6-8	-	-	2	-	-	-	2	-
MO	3-5	5, 8	-	3-8	4-5	6-8	-	-	-
NV	3, 5-7	8	K, 7-8	5	5-6, 8	-	-	-	8
NJ^	3-8	-	-	5-8	3-8	5-8	6	7-8	7-8
NM	K-8	2, 5-6, 8	5-7	2, 4-8	1-8	2, 4, 7	3-4	4-5, 7-8	5-8
NC	2, 4, 8	4-5	4, 6	8	-	5	5	-	-
OH	1-5, 7	2, 4-5, 7-8	3-4	3, 5-8	1, 3-6	8	2, 7-8	7-8	-
OK	2-4, 7-8	2-4, 6-7	4	5, 8	3, 8	-	-	-	-
OR	2-4, 6-8	2-4	8	8	3-5, 7-8	7	-	8	2-8
SC	2, 5, 7-8	5, 7	-	2, 5-8	3-4, 7-8	3-5, 8	3-5, 8	5, 8	2
SD	2, 5-8	-	-	4-8	3, 6	-	6-8	-	-
TN	1-8	4-8	-	1-5	5-8	5-8	5-6, 8	-	5-8
TX	K-3, 5-6, 8	5	4	7-8	1-2, 7-8	-	-	7	8
UT^	K, 2-7	6	7	K, 5	5-6	7	4	-	6
VT	3-8	6-8	-	3-8	K-8	6-8	K-8	-	-
VA	K-2, 4-5, 7-8	7	3, 8	5-6	7-8	7	4-5,7	-	-
WA	2, 5-8	-	6-8	2-8	2	8	5,7	-	6,8
WV	1-2, 4, 8	-	5, 8	-	5, 8	-	-	8	7
WY	3-5, 7-8	4-6	-	3-8	-	7-8	-	-	-

* DD stands for Department of Defense Education Agency; DC stands for the District of Columbia.

^ Colorado and New Jersey do not have GLEs for kindergarten through grade 2 and for kindergarten through grade 1, respectively; Utah does not have GLEs for Grade 8 only.

We find that sometimes it is not clear what a particular reasoning GLE in the state standards requires students to do. Various levels of specificity and clarity are evident in the reasoning GLEs of the state standards. Some GLEs are very specific and simple; others are simple but vague. "Predict which of two events is more likely to occur if an experiment is repeated" (Virginia, grade 2, data analysis and probability) is an example of the former while "Analyze and interpret data (prediction, inference, conclusion, etc.)" (Arkansas, grade 4, data analysis and probability) is an example of the latter. The second GLE is too broad and general, not specific enough to know what is required of students even though it is addressed in a specific content strand.

In general, lack of clarity and specificity of GLEs increases the difficulty that teachers may have when interpreting and incorporating those GLEs in the classroom. In particular, when GLEs include reasoning, the quality of reasoning students engage in will be influenced by how teachers interpret GLEs, how they enact GLEs, how comfortable they are with reasoning as well as how they promote and incorporate reasoning in the classroom. Ambiguous expectations may also cause teachers' reluctance to encourage reasoning through their mathematics teaching.

Fourth, reasoning GLEs need to be integrated in content strands. It is not likely that teachers incorporate reasoning GLEs that are not explicitly connected to content areas because it is quite challenging to implement such GLEs in the teaching of a specific topic. Moreover, in this circumstance such reasoning GLEs are not likely to be assessed on state assessments. Our overall findings indicate that state standards have difficulty integrating reasoning in their GLEs. In particular, state standards with a separate reasoning section are not likely to specify reasoning GLEs in content strands. In this case, reasoning GLEs tend to be broad and general, and isolated from specific content, such as "formulate conjectures and discuss why they must be or seem to be true." Since such GLEs are not content-specific, it may be difficult to incorporate them when teaching a particular content and topic at the classroom level. Therefore, it is suggested that state standards embed reasoning GLEs in the content strands. This will increase the clarity and specificity of GLEs as well.

Additional Issues to Consider

In addition to the expectations that we can have from state standards, there are also some other issues that need to be considered in order to incorporate the reasoning expectations at the classroom level and to change classroom practices with regard to reasoning. We describe three of those issues below.

First, to promote mathematical reasoning in the classroom, appropriate assessment tools are required. It is noted throughout the examination of the state standards that reasoning expectations are not prevalent in many states. It is surmised that one plausible explanation for this is the difficulty and expense entailed in assessing reasoning. Reasoning statements are not considered correct or incorrect, rather these responses are evaluated based on the student's ability to defend or refute their thinking with plausible arguments. Assessing reasoning requires a teacher's in-depth knowledge and understanding of the mathematical concepts. Additionally, for the most part, state assessments are typically multiple-choice items. Not only is it hard to construct items to assess student reasoning, but also it takes time, personnel, and a greater cost to score. In other words, it is not easy to measure reasoning in a large-scale assessment. Assessment tools and programs at the local and state levels should be designed to incorporate reasoning aspects as stated in state standards.

Second, reasoning should be considered one of the aspects of a student's learning progress. Historically, schools rarely communicate students' progress in reasoning to parents. Teachers need to make a commitment to not only assess reasoning in the classroom, but also communicate students' growth in the area of mathematical reasoning. School culture also needs to embrace reasoning as an essential component of mathematics education and progress.

Third, in order to promote mathematical reasoning comprehensively across grades, suitable teacher training is necessary. Classrooms in general do not pursue reasoning components of mathematics in a desired way (Stigler, Conzales, Kawanaka, Knoll, & Serrano, 1999; Stigler & Hiebert, 1999). Various aspects of reasoning and their relationships in particular are still relatively foreign to teachers. It requires teachers to devote time to create and reflect on carefully planned questions and follow-up prompting of ideas. In addition, allowing students the opportunity to share and discuss their thinking pertaining to a particular problem takes time and effort, which should not be dismissed as a trivial task for classroom teachers. Maintaining a level of dedication to this process requires commitment, experience, and focused and sustained professional development.

For example, some of the state curriculum standards include GLEs, such as "explains the solution strategy," which may or may not prompt reasoning and justification. These expectations have a potential to encourage students to reason and justify their thinking, but teachers may concentrate exclusively on the procedure when students are asked to explain solutions. With such GLEs, teachers' understanding of reasoning and their questioning skills will greatly influence the width and depth of student reasoning.

Conclusion

The mathematics education community has tried to improve classroom practices that influence the quality of student learning (RAND Mathematics Study Panel, 2003). There are many ways to accomplish such a goal, one of which will be establishing more clear and comprehensive sets of state standards. We believe that state standards will significantly influence classroom practices in terms of reasoning if they provide plausible sets of reasoning expectations that are coherent, clear, specific to content, and assessable, and if teachers are appropriately supported as they implement those reasoning expectations.

References

American Federation of Teachers. (2003). *Setting strong standards.* Washington, DC: American Federation of Teachers.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century —An ICMI Study* (pp. 37-52). Netherlands: Kluwer Academic Publishers.

Joftus, S., & Berman, I. (1998). *Great Expectations? Defining and Assessing Rigor in State Standards for Mathematics and English Language.* Washington, DC: Council for Basic Education.

Authors. (2006). Analysis of emphasis on reasoning in state mathematics curriculum standards. In B Reys (Ed.), *The de facto U.S. national mathematics curriculum: An analysis of state-level mathematics curriculum standards* (pp. 89-109). Greenwich, CT: Information Age Publishing.

Mullis, I., Martin, M., Smith, T., Garden, R., Gregory, K., Gonzalez, E., Chrostowski, S., & O'Connor, K. (2001). *TIMSS assessment frameworks and specifications 2003.* Chestnut Hill, MA: TIMSS International Study Center, Boston College.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education.* Santa Monica, CA: RAND.

Reys, B. (Ed.) (2006). *The de facto U.S. national mathematics curriculum: An analysis of state-level mathematics curriculum standards.* Greenwich, CT: Information Age Publishing.

Stigler, J., Conzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS videotape classroom study: Methods and findings from an exploratory research project on eight-grade mathematics instruction in Germany, Japan, and the United States.* Washington, DC: Department of Education, National Center for Education Statistics.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: The Free Press.

Stylianides, G., & Silver, E. (2004). Reasoning and proving in school mathematics curricular: An analytic framework for investigating the opportunities offered to students. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 611-619). Toronto, Canada: OISE/UT.