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Integrating NRC Principles and the NCTM Process Standards to Form a Class Learning Path Model That Individualizes Within Whole-Class Activities

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ABSTRACT:
This paper integrates principles from two recent National Research Council Reports (How Students Learn and Adding It Up) *with the NCTM Process Standards to form a Class Learning Path Model of classroom mathematics teaching that can help teachers achieve equity in mathematics learning by assisting all students to move forward within their own learning path to at least one general, mathematically-desirable, and accessible method. This model enables leaders to integrate research results from the national reports within a single equity perspective that can be used by teachers to individualize within whole-class activities. This model consists of three parts: three continuing teaching tasks that build a Year-Long Nurturing Meaning-Making Math Talk Community that enables students to move from and relate their entering informal math knowledge to formal academic math knowledge, four Classroom Learning Zone Teaching Phases used for each math topic to move all students along their own learning path, and Inquiry Learning Path Teaching that consists of seven responsive means of assistance that facilitate learning and teaching by all.*

Two recent National Research Council Reports, *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) and *How Students Learn* (Donovan & Bransford, 2005; Fuson, Bransford, & Kalchman, 2005) identified principles that summarize research about mathematics teaching and learning. The NCTM Process Standards likewise describe vital aspects of successful teaching and learning. It would be helpful for teachers and for leaders if all of these were integrated within a single framework. That is the task of this paper. We describe a Class Learning Path Model that can help teachers to achieve equity by assisting all students to move forward within their own learning path to general, mathematically-desirable, and accessible methods. This model consists of three parts: a Year-Long Nurturing Meaning-Making Math Talk Community built by the teacher via three continuing teaching tasks, four Classroom Learning Zone Teaching Phases used for each math topic, and Inquiry Learning Path Teaching that consists of seven responsive means of assistance that facilitate learning and teaching by all (see Table 1). At the Table 1 level only principles from the two NRC reports are involved. But at the more detailed levels described later in the paper, the NCTM Process Standards are included.

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TABLE 1: *Principles and Standards in Action in the Class Learning Path Model*

Part 1: Create the Year-Long Nurturing Meaning-Making Math Talk Community to achieve the overall goal: Build resourceful self-regulating problem solvers *(How Students Learn Principle 3)* by continually intertwining the 5 strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition *(Adding It Up)*

Part 2: For each math topic, use four Class Learning Zone Teaching Phases

Phase 1: Teacher draws out and works with the preexisting understandings that their students bring with them *(How Students Learn Principle 1)*

Phase 2: Teacher helps students move through learning paths and build networks of knowledge in various math domains *(How Students Learn Principle 2)*

Phase 3: Teacher helps students gain fluency with desired method(s) so everyone moves along their learning path; individual students stop using visual supports whenever they are able to do so; fluency includes being able to explain the method; practice is kneading knowledge, so reflection and explaining still continue *(Adding It Up: fluency & understanding)*

Phase 4: Teacher facilitates remembering by occasional practice with feedback and occasional discussions to relate ideas or method(s) to new topics that might relate or interfere

Part 3: Use Inquiry Learning-Path Teaching: This consists of seven Responsive Means of Assistance that facilitate learning and teaching by all:

- Engaging and Involving
- Managing
- Coaching
  - modeling
  - cognitive structuring and clarifying
  - instructing/explaining
  - questioning
  - giving feedback

These vary by phase and over the year. Students and the teacher give assistance.

In the first author's work on the NRC reports and on the CMW Research Project, a continuing focus was on balancing the extremes of the polar positions in the "Math Wars" concerning traditional and reform teaching. This is represented in all three parts of the model. The Nurturing Meaning-Making Math Talk Community relates students' initial knowledge and experiences to the formal math vocabulary, concepts, and methods. It nurtures and supports but also consistently communicates high expectations for all: all students work hard and move along their learning path. The four Class Learning Zone Teaching Phases allow student thinking to surface and be supported within the classroom but also introduce mathematically-desirable methods that students can understand and do. Students do not jump from Concrete and Slow informal methods to rote formal Current Common methods as in traditional teaching but to methods they can relate to visual supports and come to explain as well as carry out. No one continues concrete and slow or incorrect methods as in some approaches. Inquiry Learning Path Teaching also is balanced because it clarifies that teachers must do a great deal of assisting, but that students also assist. Inquiry is in the title to emphasize that the whole learning path environment is one of inquiry: all students and the teacher are continually seeking to increase their own understandings, which sometimes occurs by helping others or by listening to the Math Talk as well as by participating in it. Inquiry does not have to mean that students must be stuck only with the methods they invent. They can be helped to more-advanced methods that they can understand with the help of the meaning-making supports and the explanations of classmates.

The Class Learning Path Model is drawn from two models developed within the Children's Math Worlds Classroom Research Project. This project worked over 12 years in a wide range of Kindergarten through Grade 5 classrooms seeking balanced approaches to teaching and learning that would work in all classrooms. The classrooms included Spanish-speaking classrooms, English-speaking classrooms, classrooms with English language learners from many backgrounds, and classrooms with a variety of inclusion students with various special needs. Many of the classrooms had 30 to 37 students in them, even in the lower grades. Thus, the model applies well to the highly challenging situations that are unfortunately too typical today but also to suburban settings with smaller classes and more homogeneous students, which were also involved in the Children's Math Worlds Classroom Research Project. The model is also consistent with the results of research in urban low-achieving schools and intervention studies with a range of students (e.g., some of these are summarized in Fuson, 2003, pp. 88-90). Part 1 of the Class Learning Path Model is adapted from part of the Mathematics Equity Pedagogy (Fuson et al., 2000), and Parts 2 and 3 are extensions of the ZPD Mathematical Proficiency Model (Murata & Fuson, 2006). The ZPD Mathematical Proficiency Model draws from several aspects of Tharp and Gallimore's (1988) Vygotskiian perspective on literacy developed in working with many children from native Hawaiian backgrounds and with other kinds of English language learners from several different cultures. Therefore, core parts of the Class Learning Path Model apply to literacy as well as to math teaching.

The Class Learning Path Model uses several concepts from Vygotsky, who theorized about how the formal knowledge of a culture was passed on to new generations both in formal and in informal teaching. These concepts will be discussed as the relevant parts of the model are described. The model uses a constructivist view of learning: students and teachers each construct individual knowledge based on their own individual life experiences, though often through interactions and assisted by a more knowledgeable person.

The Class Learning Path Model describes processes and supports that allow teachers to individualize within whole-class activities. It is easy to describe ways to individualize instruction by breaking the class apart in various ways. However, these all require management skill, time, and energy as well as special individualized materials, and this approach may decrease student's productive learning time. Our model permits continual meeting of individual needs within whole-class instruction, minimizing the need for separate specialized activities. We close this paper by relating the Class Learning Path Model to the LATCH model for integrating math instruction for English learners (Garrison, Amaral, Ponce, 2006).

Teaching real students in classrooms is a highly complex task. Our Class Learning Path Model is thus also necessarily complex. Because of space limitations and to maximize the usefulness of the presentation to leaders working with teachers, the model is primarily presented in a series of tables that can be used with teachers. The text of the paper will serve to provide background and orientation to the tables. Mathematical examples are given after Part 2 is described.

THE CLASS LEARNING PATH MODEL
Part 1

Part 1 of the Class Learning Path Model identifies three continuing teaching tasks that must be carried out all year to build and maintain the classroom environment, the Year-Long Nurturing Meaning-Making Math Talk Community, within which learning by all can flourish (see the top of Table 2). The type of learning specified as desired by both NRC reports is integrated into one overall goal stated at the top of Part 1 of Table 2. The three continuing teaching tasks come from the NCTM Process Standards and *How Students Learn* Principle 1 (see Table 2). The Teaching Tasks 1, 2, and 3 are further described in Table 3, which shows how the special classroom environment created by the on-going teaching tasks enables all learners to relate their informal initial knowledge, what Vygotsky called *spontaneous concepts* that are formed in the real world informally and without explicit teaching, to the formal academic mathematical knowledge, what Vygotsky called *scientific concepts* that are structured and hierarchical and are formed in schools or other intentional teaching situations so that students become consciously aware of them and can reflect on them. This Part 1 environment includes a safe and nurturing teaching-learning community (Teaching Task 1), coherent learning support means of assistance to help everyone build meanings for the formal constructs that relate to but extend students' entering knowledge (Teaching Task 2), and a collaborative Math Talk culture that enables students to share and discuss their present understandings and to advance their understanding by input from the teacher and their classmates (Teaching Task 3). Students' informal preexisting

TABLE 2: *Principles and Standards in Action in the Class Learning Path Model*

Part 1: Use the continuing Teaching Tasks 1, 2, 3 to create the Year-Long Nurturing Meaning-Making Math Talk Community to achieve the overall high-level goal for all: Build resourceful self-regulating problem solvers *(How Students Learn Principle 3)* by continually intertwining the 5 strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition *(Adding It Up)*

Teaching Task 1: Teacher builds the nurturing teaching-learning community (*How Students Learn Principle 1* and *NCTM Process Standard: Communication*)

Teaching Task 2: Teacher creates a cognitively supportive referential meaning-focused classroom by using coherent visual, sensory-motor, linguistic, and situation supports along with math modeling to create interest and accessibility of ideas *(NCTM Process Standards: Connections & Representation)*

Teaching Task 3: Teacher develops a collaborative Math Talk (instructional conversation) culture *(NCTM Process Standards: Problem Solving, Reasoning & Proof, Communication)*

Part 2: For each math topic, use four Class Learning Zone Teaching Phases

Phase 1: Teacher draws out and works with the preexisting understandings that their students bring with them *(How Students Learn Principle 1)*

a. Teacher elicits, values, and discusses student ideas and student methods
b. Teacher identifies students who use different levels of solution methods and those who are doing typical errors and ensures that these are seen and discussed by the class

Phase 2: Teacher helps students move through learning paths and build networks of knowledge in various math domains *(How Students Learn Principle 2)*

a. Teacher focuses on or introduces mathematically-desirable and accessible method(s)
b. Erroneous methods are analyzed and repaired with explanations
c. Advantages and disadvantages of various methods including the Current Common method are discussed so that central mathematical aspects of the topic become explicit
d. Explanations of methods and of mathematical issues continue to use quantity and/or spatial language and visual supports to help all students build networks of knowledge and move along their own learning path

Phase 3: Teacher helps students gain fluency with desired method(s) so everyone moves along their learning path; individual students stop using visual supports whenever they are able to do so; fluency includes being able to explain the method; practice is kneading knowledge, so reflection and explaining still continue *(Adding It Up: fluency & understanding)*

Phase 4: Teacher facilitates remembering by occasional practice with feedback and occasional discussions to relate ideas or method(s) to new topics that might relate or interfere

Part 3: Use Inquiry Learning-Path Teaching: This consists of seven Responsive Means of Assistance that facilitate learning and teaching by all: Engaging and Involving, Managing, Coaching (modeling, cognitive structuring and clarifying, instructing/explaining, questioning, giving feedback). These vary by phase and over the year. Students and the teacher give assistance.

vocabulary, ideas, and methods form the foundation from which the teacher builds up to the higher formal mathematical vocabulary, ideas, and methods using the resources in the Nurturing Meaning-Making Math Talk Community. There is an on-going interaction between the formal and informal vocabulary, ideas, and methods (indicated by the vertical bi-directional arrow in Table 3). All three continuing Teaching Tasks involve what Vygotsky called semiotic tools: oral language, written notations and drawings, and +physical objects that facilitate student learning of concepts and their relating of formal and informal versions of these.

Levels in Math Talk that move from traditional teacher-focused talk to student-to-student talk with teacher assistance are described in Hufferd-Ackles, Fuson, & Sherin (2004); the higher Math Talk Levels 2 and 3 give space in the classroom discourse for all voices to emerge and to move forward correcting errors and increasing understanding.

TABLE 3: *Use the Continuing Teaching Tasks 1, 2, 3 to Create the Year-Long Nurturing Meaning-Making Math Talk Community as the Environment to Relate Students' Vygotskiian Informal Knowing to Formal Mathematical Knowing*

Formal mathematical vocabulary, ideas, and methods: Bring students up to the higher mathematics in meaningful ways and by small supported coherent steps

Via a Nurturing Meaning-Making Math Talk Community

Teaching Task 1: Teacher builds the nurturing teaching-learning community: Co-creates an inclusive and participatory classroom culture in which the class co-constructs emerging related understandings for all by providing multiple levels of access (everyone can participate) through mathematizing (seeing the math in children's worlds); making math drawings; using rich language by validating all children's language and experiences while connecting them to standard language and symbols; and facilitating listening, speaking, writing, and helping competencies to make problems accessible to all

Teaching Task 2: Teacher creates a cognitively supportive meaningmaking classroom by using coherent visual, sensory-motor, linguistic, and situation learning supports along with math modeling to create interest and accessibility of ideas: Mathematical words and symbols are linked to coherent meaningful referents by mathematizing known contexts or by providing new experiences to be mathematized; rich language use by all (see Teaching Task 1); everyone makes Math Drawings or uses other visual or sensory-motor supports to facilitate reflection, discussion, analysis, and understanding of everyone's thinking

Teaching Task 3: Teacher develops a collaborative Math Talk (instructional conversation) culture of understanding, explaining, questioning, justifying, and helping that elicits, values, and discusses student ideas and methods while relating visual quantities to steps in each method and discussing mathematical attributes of methods; talkers and listeners can understand each other because Math Talk connects to referents (see Teaching Task 2); all teachers are learners and all learners (students) are teachers of themselves and of others (peer helping); all participants help to develop coherent networks of knowledge by relating ideas and experiences within instructional conversations (Math Talk)

Informal preexisting vocabulary, ideas, and methods: Start where students are and keep learning meaningful

Note. The vertical arrow indicates that the formal and informal vocabulary, ideas, and methods continually relate to each other via the Teaching Tasks 1, 2, 3.

Table 4 shows an abbreviated version of the table in Hufferd-Ackles et al. with a full description of the highest level. The term *instructional conversation* was used by Tharp and Gallimore (1988) to emphasize that the talk has learning purposes and should move participants (including the teacher) forward in their own learning paths and that it is not a teacher lecture. We included the term to emphasize that Math Talk involves students but is led by the teacher toward mathematical learning goals.

Part 2

Part 2 of the Class Learning Path Model appears in the middle of Table 2. The four Class Learning Zone Phases reflect Vygotsky's cultural model of teaching in which assistance from others, and language and actions during such assistance, eventually became assistance provided by the self, first externally and then, especially with language, are internalized into internal speech. This movement from other- to self-assistance occurred within what Vygotsky called the Zone of Proximal Development (ZPD): *the distance between the actual developmental level as determined by individual problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers.* (Vygotsky, 1978, p. 86). Each child has an individual Zone of Proximal Development (ZPD) for each kind of learning topic. However, when the four Class Learning Zone Teaching Phases are carried out within the Year-Long Nurturing Meaning-Making Math Talk Community, the whole class is working within a Class

Learning Zone and creating a Class Learning Path within which a limited number of solutions methods ranging from concrete and slow to advanced are described by students (and sometimes by the teacher) and discussed and related to each other. Within this Class Learning Zone everyone moves forward on their own individual learning path within their own zone of proximal development. Because these individual learning paths are related mathematically, Math Talk about different related methods can help everyone progress.

The first two phases in Part 2 (see Table 2) come from *How Students Learn Principles 1 and 2*. They describe how the teacher begins by eliciting student thinking and then begins to move along a Class Learning Path by focusing on or introducing mathematically-desirable and accessible methods and analyzing and repairing erroneous methods. Phase 3 comes from the *Adding It Up* focus on both understanding and fluency. Visual supports for understanding are dropped when an individual student no longer needs them, but fluency includes being able to explain a method and relate it to a visual or situational support. Phase 4 comes from the ZPD Mathematical Proficiency model (Murata and Fuson, 2006) as well as from basic learning research that indicates that occasional practice with feedback is required for a long period of time for new learning to be remembered effectively. The relational nature of mathematics also means that new related topics will arise that provide opportunities to re-view the original topic by relating it to the new topic.

Both NRC reports summarized and drew upon for their principles the explosion of worldwide research about student thinking in various math topics. This research indicates that students will be able to discuss their own ideas about a math topic that is presented in some meaning-making context or with some learning support. Therefore, Phase 1 is fruitful (will lead to student ideas and methods) if it occurs within a Nurturant Meaning-Making Math Talk Community (Part 1 of the model). This same research indicated how student methods for many topics fall into a learning path of increasing abstractness and abbreviation that move from concrete and slow methods to faster methods, some of which are general and accessible. Research from around the world also indicates that different solution methods are taught in different countries. These methods are often complex and abbreviated and thus relatively difficult to learn with meaning. In the United States these are often called "the standard algorithms," but *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) stressed that this term is misleading because different algorithms have been taught at different times in this country. We therefore call these methods the Current Common methods. Research has identified instead algorithms and other kinds of solution methods that are mathematically-desirable and more accessible (MD & A) to students than are the Current Common methods. These more accessible methods fit students' thinking better, so they are easier for students to understand and to explain. Most are easier to do procedurally and are less prone to errors than are the Current Common methods. But each clearly uses at least one important mathematical idea and so is a worthy focus of Math Talk that will make this idea clear to students. Some of these methods are described in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001), in the research volume accompanying the NCTM Standards 2000 (Fuson, 2003), and in Fuson (2006).

These mathematically-desirable and accessible methods are what enables instruction to be differentiated within whole-class activities when they are taught with all three parts of the Class Learning Path Model. The Nurturant Meaning-Making Math Talk Community enables all children in a class to understand at least one of the mathematically-desirable and accessible methods when it is taught with the seven means of assistance that constitute the Part 3 Inquiry Learning Path Teaching (to be discussed shortly). Table 5 shows how the differentiated learning works within the four whole-class phases. Each student does advance within his/her own learning path. But what makes things manageable within the whole-class context is that, for any given math topic, there are a limited number of methods that students develop and share and there are also a limited number of kinds of errors made by students. So it is possible to share the range of student methods within the Nurturant Meaning-Making Math Talk Community and to surface and address the errors within the Math Talk. The coherent learning supports introduced for the topic enable the Math Talk to be comprehensible to all listeners.

Phase 1. In Phase 1 methods are elicited from students. These (see Table 5) include incorrect methods, concrete and slow methods, general and accessible methods, and sometimes the Current Common method which is identified in Table 5 as a student method if it is introduced by a student rather than by the teacher. This process allows all cultural methods taught at home to be voiced in the classroom, where they can be explained with the help of the

TABLE 4: *Levels of Math-Talk Learning Community: Teacher and Student Action Trajectories*

Components of the Math-Talk Learning Community			
A. Questioning	B. Explaining math thinking	C. Source of math ideas	D. Responsibility for learning
Overview of Shift over Levels 0 - 3: The classroom community grows to support students acting in central or leading roles and shifts from a focus on answers to a focus on mathematical thinking.			
Shift from teacher as questioner to students and teacher as questioners.	Students increasingly explain and articulate their math ideas.	Shift from teacher as the source of all math ideas to students' ideas also influencing direction of lesson.	Students increasingly take responsibility for learning and evaluation of others and self. Math sense becomes the criterion for evaluation.
Level 0: Traditional teacher-directed classroom with brief answer responses from students.			
Level 1: Teacher beginning to pursue student mathematical thinking. Teacher plays central role in the math-talk community.			
Level 2: Teacher models and helps students build new roles. Some co-teaching and co-learning begins as student-to-student talk increases. Teacher physically moves to side or back of the room and directs from there.			
Level 3: Teacher as co-teacher and co-learner. Teacher monitors all that occurs and is still fully engaged.			
Teacher is ready to assist, but now in more peripheral and monitoring role (coach and assister).			
Teacher expects students to ask one another questions about their work. The teacher's questions still may guide the discourse. The teacher's questions still may guide the discourse. Student-to-student talk is student initiated, not dependent on the teacher. Students ask questions of each other and listen to responses. Many questions are "Why?" questions that require justification from the person answering. Students repeat their own or other's questions until satisfied with answers.	*Teacher follows along closely to student descriptions of their thinking, encouraging students to make their explanations more compete; may ask probing questions to make explanations more complete. Teacher stimulates students to think more deeply about strategies.* Students describe more complete strategies; they *defend and justify* their answers with little prompting from the teacher. Students realize that other students will ask them questions, so they are motivated and careful to be thorough. Other students support with active listening.	*Teacher allows for contributions from students during her explanations; she lets students explain and "own" new strategies. (Teacher is still engaged and deciding what is important to continue exploring.) Teacher uses student ideas and methods as the basis for lessons or miniextensions.* Students contribute their ideas as the teacher or other students are teaching, confident that their ideas are valued. Students spontaneously compare and contrast and build on ideas. Student ideas form part of the content of many math lessons.	*The teacher expects students to be responsible for co-evaluation of everyone's work and thinking. She supports students as they help one another sort out misconceptions. She helps and/or follows up when needed.* Students listen to understand, then initiate clarifying other students' work and ideas for themselves and for others during whole-class discussions as well as in small group and pair work. Students assist each other in understanding and correcting errors.

meaning-making supports in the classroom and related to other methods.

Phase 2a. In Phase 2a the mathematically-desirable and accessible methods are introduced by the teacher or by the math program (e.g., as methods used by characters in a story or students in someone's class), again linked to the visual and other meaning-making learning supports. Because of the visual and verbal learning supports in the Math Talk Community, one of these methods can be learned by each student in the class, in contrast to the current common method, which is more complex and abstract. We included two mathematically-desirable and accessible methods in Table 5 because research has identified in many areas two such methods that vary in the mathematical attributes they emphasize. Introducing both

TABLE 5: *Differentiated Learning Within the Class Learning Zone Phases: Everyone Advances Within Own Learning Path*

Phase 1: Students enter with a range of methods ranging from concrete and slow to advanced and rapid; some may know the current common method [CC], which is labeled student method A below if it is demonstrated initially by a student.

Phase 2a: Teacher focuses on or introduces mathematically-desirable and accessible method(s) [MD&A] and ensures that erroneous methods are analyzed and repaired with explanations.

Phase 2b: Teacher introduces current common method [CC] if it has not already been demonstrated by students, and students relate it to MD&A method(s) during Math Talk.

Phase 3 & 4: Students become fluent in one mathematically-desirable and accessible, general and accessible, or current common method; many students become fluent in two or three such methods. Students maintain or finally achieve fluency by occasional practice with feedback and occasional discussions to relate ideas or method(s) to new topics that might relate or interfere.

Type of Method	Phase 1	Phase 2a	Phase 2b	Phases 3 & 4
Current Common [CC]	student method A?	student method A?	**CC method related to MD&A methods**	CC method?
Mathematically-Desirable & Accessible [MD&A]	student method B?	**MD&A method a** **MD&A method b**	MD&A method a MD&A method b (may be abbreviated)	MD&A method a MD&A method b (may be abbreviated)
General & Accessible	student method C	may continue	may continue	may continue
Concrete & Slow	student method D	**move on to MD&A method a or b**		
Incorrect	student method E	**Discuss and repair errors in methods**	**Monitor; repair if reappear**	**Monitor; repair if reappear**
Number of methods by one student	most 1 or 0	most 1, some 2 or 3	all 1 MD&A or G&A or CC many 2 or 3 methods	all 1 MD&A or G&A or CC many 2 or 3 methods

Note. ? means that this method may not be used by any student. More than one student method of a given type may be used.

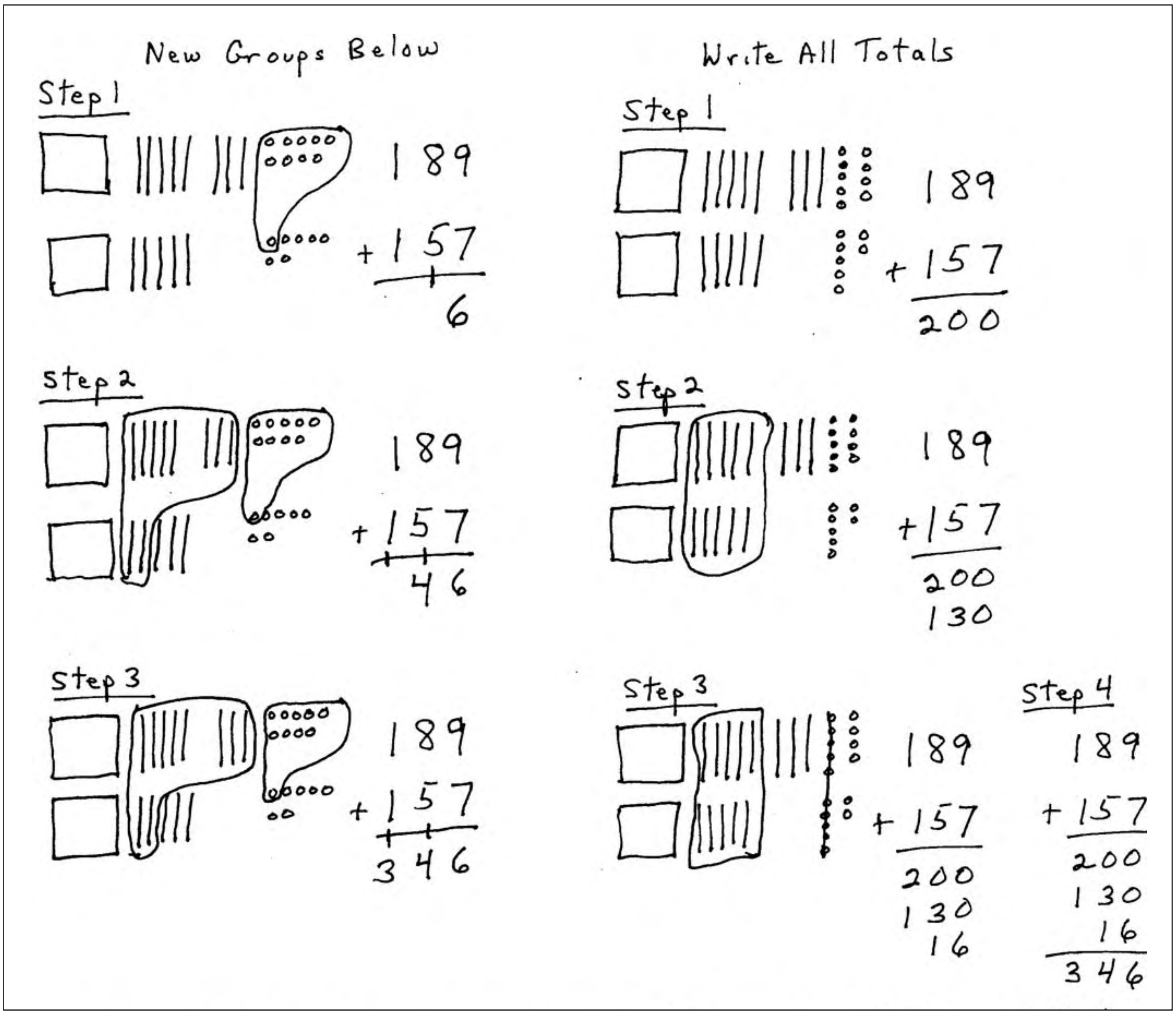
permits fuller understanding of the math topic even for those students who learn only one of the methods. They may vary in abstractness so that less-advanced students choose the more concrete or visual method, or they may just appeal to individual differences in students (Fuson, 2006). In all explanations in all phases, it is important to link the math drawing or other visual support to the formal math method for each step of that method. It is such tight linking that enables the meanings for the visual or contextual supports to become attached to the formal math method and notations and thus to take on those meanings.

During Phase 2a all students experience and discuss advantages and disadvantages of the mathematically-desirable and accessible methods. Students who were using concrete and slow methods are asked to choose one of the methods, become fluent in it, and become able to explain its steps using meaningful standard mathematical language. Such explanations are given first by more-advanced students and clarified and extended as needed by the teacher, so these less-advanced students hear examples before they explain themselves. Students who were using a general and accessible method (or a Current Common method) may continue using their method as long as they can explain it linked to the visual quantity support being used for that topic; the teacher and other students assist with such explanations. Errors also continue to be discussed and repaired. By the end of Phase 2a almost all students have moved from concrete and slow and from incorrect methods to a mathematically-desirable and accessible method. Some students enjoy trying all of the methods that have been introduced (all those beyond the concrete and slow methods) and may vary the method they use for different problems.

Phase 2b. In Phase 2b the Current Common method is introduced by the teacher if it has not already been demonstrated by a student and is related within Math Talk to the mathematically-desirable and accessible methods. Such methods are chosen to relate easily to the Current Common method so that Math Talk is accessible and so that parents who know the Current Common method can readily understand the mathematically-desirable and accessible methods.

Phase 3. In Phase 3 students build fluency for their chosen method or methods. No student is now using a concrete and slow method, and errors have been greatly reduced. Some less-advanced students may still be making math drawings, but many students no longer are. Math Talk explanations continue to enable all students to build or strengthen their network of knowledge for the topic as well as increase their fluency for their method(s). We found that in many classrooms the majority of students during Phase 3 enjoyed mastering and using two or three methods. The mathematically-desirable and accessible methods are

FIGURE 1. ***Linked drawing and numerical steps for addition methods.***



chosen so that they are rapid enough to be used for life. Thus, they do not have to be replaced by the Current Common method, though they may be whenever a student so chooses and can explain the Current Common method (this keeps the emphasis on understanding as well as on fluency).

Phase 4. Phase 4 is important because sometimes errors can creep back in, especially for older students who have been using an erroneous method for a year or more before learning a mathematically-desirable and accessible method (e.g., subtracting the smaller from the larger number even if the larger number is on the bottom is an extremely widespread error at all grades and even into high school). During this phase it is often enough to ask students to think about a math drawing (or other visual support) for them to be able to correct the error.

Examples of the Types of Methods

Multidigit addition. Examples of mathematically-desirable and accessible methods for multidigit addition are shown in Figure 1 along with math drawings of the hundreds, tens, and ones that students would make to support their understanding and explanations of their numerical methods. These methods were in *Adding It Up* Kilpatrick, Swafford, & Findell, 2001; p. 202) and are discussed more fully in Fuson (2006). The first method (New Groups Below) is just like the Current Common method (which could be called New Groups Above, see both of these methods in Figure 2) except that the new group (new ten, hundred, thousand, etc.) is written below the next left column on the line rather than above it. The New Groups Below method generalizes to any number of places and has three advantages over the Current Common (New Groups Above) method:

a) When you write the new group below, it is near the ones of the teen number you made, so you can see the whole teen number more easily. This clarifies what you are actually doing when you make the new group and put it in the next left column. For example, when adding the 9 ones and 7 ones, you can see the 16 much more easily in New Groups Below (see Figure 1) than in the New Groups above method where the 1 and the 6 are separated so far apart.

b) It is much easier to add the numbers whenever you have a new group because you add the two numbers you see in the problem (e.g., 8 tens and 5 tens in Figure 1) to get 13 tens and then add the 1 ten waiting below to get 14 tens. In the Current Common (New Groups Above) method (see Figure 2), you add the 1 to the top number 8, hold that total 9 in your mind while you add to it the bottom number 5 (you can't even see the second number 9 and you can see the old top number 8 that you are no longer using).

c) Some students object to the Current Common (New Groups Above) method, saying that you are changing the problem when you put the 1 up there. And actually you are changing the addition problem when you do that. For example, the 1 new ten above the 8 tens in Figure 2 changes the top number from 189 to 199. In New Groups Below the new 1 group stays down below in the answer space not changing the problem.

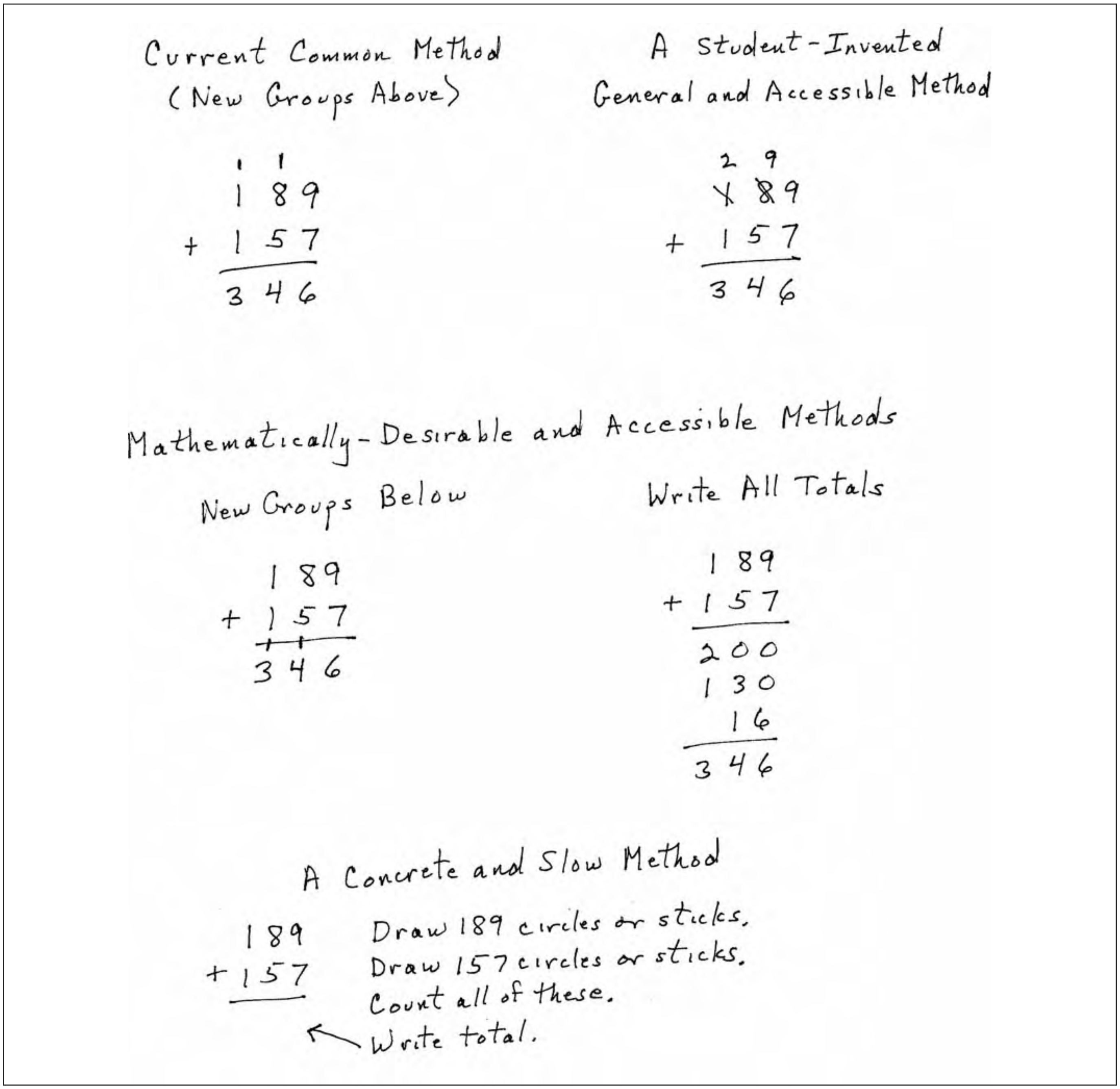
The second research-based mathematically-desirable and accessible method, the Write All Totals method (see Figure 1), shows the total of each place value written using all needed zeroes. This method can go from the left (shown in Figure 1) or from the right (the rows of subtotals would just be reversed). Most students prefer to go from the left; teachers of special needs students find this method valuable.

The Write All Totals method eventually becomes cumbersome for very large problems, but is worth introducing and discussing to help less-advanced students see the values they are adding for numbers in the millions. Seeing both New Groups Above and New Groups Below in many places helps students understand the generality of making 1 new group of the next larger multiunit from ten of the smaller units to the right. The 3 advantages of New Groups Below continue for such large numbers.

The New Groups Below and Write All Totals methods generalize to decimal positions to the right of one. The Write All Totals method helps students see how to add thousandths, hundredths, and tenths and to verbalize that 10 thousandths make 1 hundredth and 10 hundredths make 1 tenth; these are initially difficult because the verbal patterns are in the opposite direction to those for whole numbers where 10 hundreds make 1 thousand. For these places we use dimes, pennies, and a picture of a sectional tenth of a penny to help students visualize and remember that the places are getting smaller as you move to the right (they are getting one-tenth as big).

Figure 2 shows the above three methods and also a student-invented general and accessible method and a concrete and

FIGURE 2. ***Methods for multidigit addition.***



slow method. The former is a variation of the Current Common (New Groups Above) method in which the one new ten or hundred is added into the top number rather than being written above ready to add in. This method was invented by students using base-ten blocks, who added the new ten or hundred in with the blocks for the top row (Fuson & Burghardt, 2003). This method could also be done with math drawings such as shown in Figure 1. It has the same first advantage over the Current Common method as does New Groups Below: the addition for each column is easier. This method is general (it can be extended to larger whole and to decimal numbers), and it is accessible to students. In the Children's Math Worlds Project we introduced students to New Groups Below rather than this method because of the latter's three advantages and because some students confused this method with subtraction because of the crossing out of top numbers.

A concrete and slow method is to make a drawing of things (or circles or sticks) for each number and count all of the things by ones. Many students do invent such a method for 2-digit numbers; even for such numbers it is very slow and often inaccurate. Students who learn quantity drawings that show hundreds, tens, and ones such as in Figure 1 have no need to do such a slow method and can immediately understand and use one of the mathematically-desirable and accessible methods. Students may also use methods that count on by hundreds and tens; these are not general (they become very awkward even for large hundreds) and are not accessible (many students do not have those skills of counting on, and they take time to develop).

These examples show all of the types of methods listed in Table 5. The Current Common method for addition is relatively accessible to students. The Current Common methods for multidigit subtraction, multiplication, and division are less so. Introducing mathematically-desirable and accessible methods such as those shown in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) and in Fuson (2006) can be very helpful in allowing all students to move up to such a method that they can understand, do, and explain.

The importance of math drawings. In the Children's Math Worlds Classroom Research Project we found that moving as rapidly as possible in each topic to having students make math drawings along with their solution methods was extremely powerful in supporting everyone in the Math Talk Community to understand and participate in the instructional conversation. Math drawings focus on the mathematical aspects of a problem and are as simple as possible (e.g., for a word problem about cats, children draw circles rather than pictures of cats). Math drawings can be made rapidly on the class board, on class activity sheets, and on homework. They help English language and less-advanced learners follow the Math Talk. Non-English speakers can gesture to parts of their math drawing and then to their numerical or geometric solution to relate these, and a helping classmate can voice their explanation, checking with the explainer that it is correct. Many students, even native English-speakers, usually can comprehend more than they can say, but this process of explaining using a math drawing allows them to participate before they have become fluent in the formal math English needed for a full explanation.

One can see the power of math drawings by looking at the quantity math drawings shown in Figure 1. The drawings themselves were taught as part of Teaching Task 2 when students were discussing place value. The vertical ten-sticks were originally ten circles connected by a vertical stick. The hundred-boxes originally contained 10 ten-sticks. Variations in math drawings come from individuals and are not associated with any particular numerical method. Students link the drawings step-by-step to each mathematically-desirable and accessible numerical method and explain each step in their drawing linked to that step in their numerical method. They must use quantity language (hundreds, tens, ones) when adding tens or hundreds. For example in New Groups Below, they say "eight tens plus five tens is thirteen tens plus one more ten waiting here below is fourteen tens, which is one hundred and four tens." Or they may say "eighty plus fifty is one hundred thirty (they can see in the drawing how 8 tens need 2 more tens to make 10 tens, which equal 1 hundred) plus one more ten from the ones makes one hundred forty." They do not say "eight plus five" when adding tens or hundreds. This quantity language helps the numerical method to take on these quantity meanings, which will remain when students no longer need to make the drawings. They now can make the verbal quantity explanations when looking only at the numerical method.

Students vary in how they make the new 1 ten or new 1 hundred from the ones or from the tens. Each such variation supports different mental single-digit methods, which are also facilitated by the 5-groups in the drawings. The top left drawing shows in the ones that 9 needs 1 more to make ten; when that 1 is taken from the 7 it becomes 6, so 1 ten plus 6 equals 16. The middle left drawing shows the same make-a-ten method for adding tens, but here the 8 tens need 2 tens taken from the 5 tens to make 10 tens, which leaves 3 tens. Step 3 on the right shows the 5s within the 9 and the 7 added to make 1 ten, leaving the 4 in the 9 and the 2 in the 7 to be added to make 6. So as students explain their methods and their classmates see variations in their drawings, different more-advanced mental methods for single-digit addition are also supported.

Single-digit subtraction. Methods of single-digit subtraction that move from the concrete and slow Take Away method to the mathematically-desirable and accessible Count Up and Make a Ten methods to the current common Recall/Memorize method are shown in Figure 3. *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) summarized the massive world-wide research literature on the developmental/experiential levels in single-digit addition and subtraction methods found around the world (see also Fuson, 1992, 2003). These levels move from

a) early conceptual structures in which students are able to consider only one number at a time: to Take Away for 14 – 8 = ?, they first make 14, then take away 8, then count the rest as 6; to

b) an embedded number concept in which an addend is embedded within the total: to Count Up, 14 – 8 = ? is thought of as 8 + ? = 14 and the 8 is embedded within the 14: "9, 10, 11, 12, 13, 14; that's 6 more from 8 to make 14." to

c) derived fact strategies in which students can make chunks within addends to use a known problem into an unknown problem: for Make a Ten, 10 + 4 = 14 is used to find 8 + ? = 14: 8 + 2 makes 10 plus 4 more in the 14 makes 6.

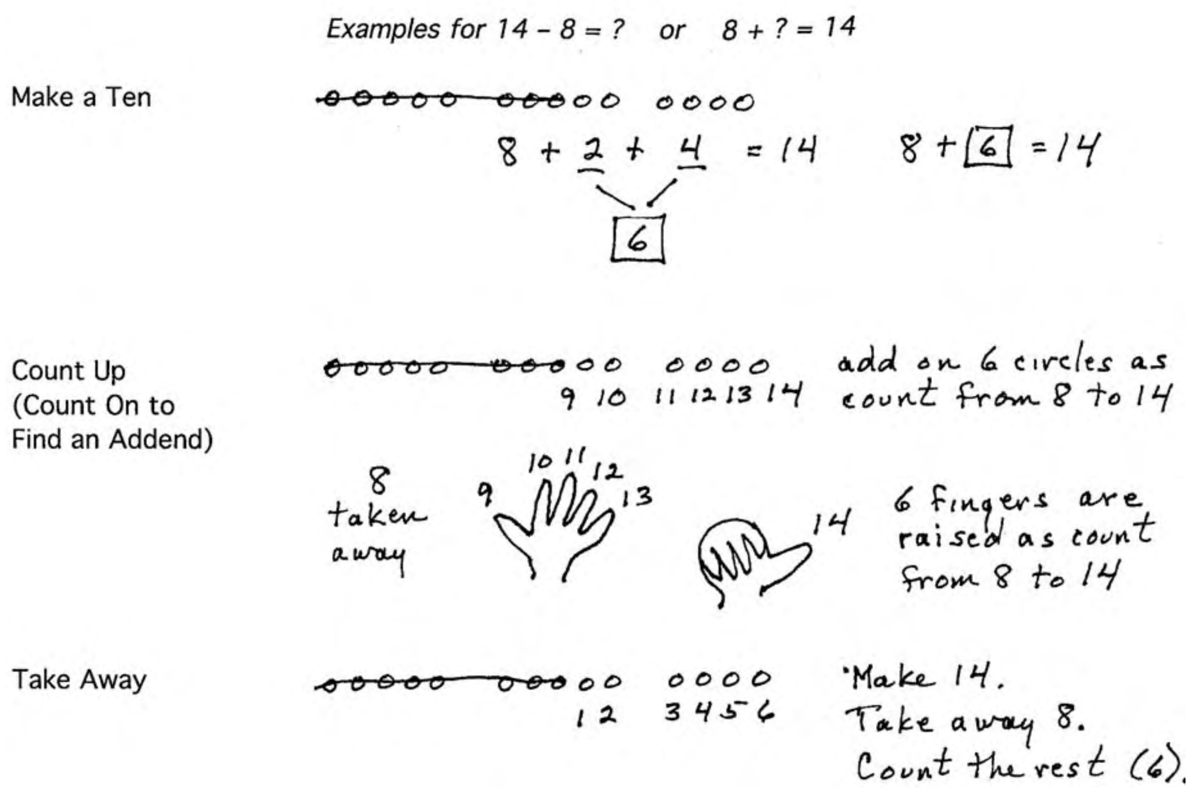
Many countries around the world help students move from the concrete and slow informal Take Away method to the mathematically-desirable count up method, and some also help students to move on to the more-advanced Make a Ten method. This method is particularly valuable in multidigit subtraction, where ungrouping gives a top number that is a ten from the next left column and the number in the top column. In Korea, the multidigit subtraction algorithm taught is to write the new ungrouped10 above the column to the right because that facilitates the make-a-ten single-digit subtraction for that column (Fuson & Kwon, 1992). For example, if 4 is in the top ones place and an 8 is below to subtract, after ungrouping a 10, a Korean child would see that 4 and a 10 above it, so could easily do Make a Ten by thinking from 8 to the 10 they see is 2 and 4 more in the 4 they see makes 6. In the U. S. it is typical to show that top number as a teen number, i.e., to write a small 1 to the left of the top number or cross out and rewrite the whole top number as 14 (we did the latter because it is clearer). In this case one can still do the Make a Ten method to find 14 – 8, but the 10 is not there visually as a support.

In the United States many students invent and use a counting down method. But these methods are difficult to carry out, and many students make errors in carrying out this method. Students in fact use four different counting down methods used, two of which are systematically wrong (Fuson, 1984). You can start counting down with the total, and then the unknown addend will be one less than the number you say when you've counted down the known addend. For example, for 14 – 8, "14, 13, 12, 11, 10, 9, 8, 7 (counted down 8, usually kept track of by raising 8 fingers as you count), so there are 6 (the next number down fom 7) left." Or you can start counting down one less than the total, and the last number counted down is the answer: "13, 12, 11, 10, 9, 8, 7, 6 (counted down 8), so there are 6 left." But students do both incorrect combinations of these, yielding an answer one too big (start with 14 and give the 8th word said as the answer: 7) or one too small (start with 13 and give the number following the 8th word said as the answer: 5).

Learning to solve subtractions by forward methods (i.e., as 8 + ? =14) has two major advantages: the forward methods make subtraction as easy as addition, and they emphasize the relationships between addition and subtraction. Addition is finding an unknown total, and subtraction is finding an unknown addend. For this reason (and to simplify terminology), in the Children's Math Worlds Research Project we distinguished adding counting on from subtracting counting on (also called counting up) by calling the adding method Counting On to Find the Total and the subtracting method Counting On to Find an Addend. The keeping track process for subtracting is easier than that for adding because you just stop when you hear the total and then look at your fingers to see the answer. For adding you must monitor your fingers until you see the second addend you are counting on. Similarly, Make a Ten to subtract is easier than Make a Ten to add because for the former, you need only find the amount to make ten with the known addend (e.g., 8 + 2 = 10) and then add that amount to the ones number you see in the teen total (add 2 to the 4 in 14). For adding Make a Ten, you need to separate the second addend into the amount to make ten (the same first step as in adding) and then find the rest of that second addend to make the ones place in the teen total: 8 + 6 is 8 + 2 + ?; think 2 + ? = 6, so 4, so 10 + 4 is 14. This is easier in East Asian languages where 14 is said as "ten four" so students do not have to know the extra European teen language step of knowing that 10 + 4 is "fourteen = 14." Very low Taiwanese students learn to use make-a-ten for subtraction before they can do so for addition (Duncan, Lee, & Fuson, 2000). In the CMW Project we found that first graders of all levels can learn Counting On to add and to subtract, and some/many also can use the Make-a-Ten methods. Others began to use the Make-a-Ten methods during multidigit addition and subtraction. Still others remained with counting on for single-digit adding and subtracting. As with all mathematically-desir-

FIGURE 3. ***Methods of single-digit subtraction.***

Current Common	Recall/Memorize
Mathematically-Desirable & Accessible	Make a Ten Count Up (Count On to Find an Addend)
General & Accessible	none (Count Down is not accessible and is often inaccurate)
Concrete & Slow	Take Away
Incorrect	Count Down is frequently incorrect



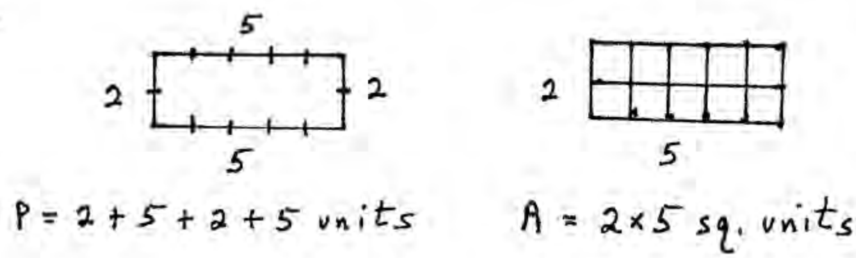
able and accessible methods, this is rapid and accurate enough to be used in any more complex problem solving and thus does not have to be replaced.

The conceptual way to enable students to move through the worldwide levels shown in Figure 3 and relate subtraction to addition is to show subtraction using 5-structures and 10-structures within math drawings such as are shown in Figure 3 and to take away the first objects rather than take away from the right. So for all three levels in Figure 3, the methods build on each other (e.g., you can see that 2 more from 8 makes 10 and there are 4 more in 14). All these methods show taking away 8, and you can even start Count Up and Make a Ten by saying “8 taken away” and continuing with the method.

FIGURE 4. ***Methods of finding the perimeter of a rectangle.***

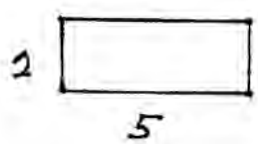
Current Common	$P = 2 \times (l + w)$
Mathematically-Desirable & Accessible	$P = 2(b + h)$ $P = 2 \times (b + h)$
General & Accessible	$P = 2l + 2w$ $P = l + w + l + w$
Concrete & Slow	count all length segments around the edge
Incorrect	$l + w$ because only those numbers appear on the rectangle shown for the problem

Meaning of perimeter versus area



Misleading perimeter problem (only 2 sides shown)

Find the perimeter.
Find the area.



The current common method Recall/Memorize is of course useful for smaller additions and subtractions (most of the totals below ten). Students use this method from the very beginning (e.g., for 1 + 1) of their addition/subtraction experience, and they continue to solve new unknown totals or unknown addends by this method. But especially for totals between 10 and 18, the Make a Ten and Count Up (Count On to Find an Addend) methods are fast and accurate enough for all purposes, and their use can even reduce the interference between addition and multiplication memorized facts that interferes with multiplication learning. State standards should reflect the massive worldwide research on these levels and require students to be fast and accurate with single-digit addition and subtraction rather than specify the method by which they must demonstrate such fluency (only memorized or recalled "facts").

Perimeter of rectangles. Methods for finding the perimeter of a rectangle are shown in Figure 4. Math drawings that show the meanings of perimeter and area are shown below. Students initially need experiences drawing rectangles using inches and centimeters to experience different measure units in use in perimeter and area. Such experiences can help them see the unit lengths of inches or of centimeters so that these, rather than the endpoints, are the units that are counted to make the perimeter.

The vital visual/conceptual points to make are that perimeter is the total of the *length units* all of the way around the rectangle and area is the total of the *square units* that cover the surface of the rectangle. Because perimeter problems are typically shown with a rectangle that has numbers for only two adjacent sides, a common incorrect method for finding perimeter is to add only those two numbers shown rather than also adding in or otherwise using the other two sides. In the CMW Research Project, we found that it helped students understand both of these points if they made two small math drawings for such a problem (see the next to bottom row in Figure 4). For perimeter, they marked and labeled the length units all around the rectangle and wrote the perimeter as the sum of all four sides. For area, they drew a second rectangle, drew in the grid of square units, and wrote the area product. Students stopped making such drawings whenever they no longer needed them.

The conceptually most accessible methods for perimeter move from the concrete and slow informal method of counting all of the length units around the sides of the rectangle (the basis for understanding what perimeter is) to general and accessible numerical methods of adding the length units rather than counting them (see Figure 4). Students invent the latter methods once they understand what perimeter is. The current common method emphasizes that one only needs the lengths of adjacent sides by using the more-advanced but less-accessible formula "the sum of the length and the width taken two times." The mathematically-desirable and accessible methods are variations of this current common method that use *base* and *height* instead of *length* and *width* in order to relate rectangles to parallelograms and triangles, where *base* and *height* instead of *length* and *width* are used. This also avoids the ambiguities in the terms *length* and *width* (is the *length* the *base* or is it the longer side?). Of course, students as always need to be introduced to the current common method and to vocabulary it uses (the terms *length* and *width*).

Part 3

Based on Vygotsky's concept of the Zone of Proximal Development (ZPD) within which a learner is assisted by more knowledgeable others, Tharp and Gallimore defined teaching as follows: *Teaching can be said to occur when assistance is offered at points in the ZPD at which performance requires assistance.* (1988, p. 31). Tharp and Gallimore identified 6 means of assistance used in teaching. We identified in the ZPD Mathematical Proficiency model (Murata & Fuson, 2006) one more, resulting in the 7 means of assistance that constitute Inquiry Learning Path Teaching, Part 3 of the Class Learning Path model (see Table 1). These means of assistance are used within the Part 1 Nurturing Meaning-Making Math Talk Community and throughout the Part 2 four Class Learning Zone Teaching Phases. Both the teacher and students assist learning. A vital role of the teacher all year is to assist students in learning how to assist better, and all students can improve in such assisting. However, we found in the Children's Math Worlds Research Project that that even some first graders and kindergarten students are natural assisters without such teacher help so that the Math Talk Community has an initial basis of assistance from classmates as well as from the teacher.

There are three main categories of responsive assistance: Engaging and Involving, Managing, and Coaching. Engaging and Involving is important throughout the four phases but is especially critical at the beginning of a new topic where some students may feel overwhelmed. Managing by students of course must be set up by the teacher, but students can take over substantial aspects of managing materials and student movement if the teacher assists them to learn to do so. The five Coaching means of assistance in Table 1 are ordered from the most to least structuring done by the assister: modeling, cognitive structuring and clarifying, instructing/explaining, questioning, giving feedback.

The word *responsive* is crucial for the means of assistance. This means that assistance is only given to individuals when they need it and at the points at which they need it. Doing more creates dependence. The Math Talk Community permits assistance to be given individually, but the teacher and classmates must learn to give long wait times while students attempt to explain before jumping in to help. Our CMW teachers called this "biting their tongue" and stressed that it was initially difficult to do; they were used to doing most of the talking in the classroom. However, when they did leave space for student voices to emerge, and managed the class from the side or back of the room during Math Talk, they were frequently impressed by what their students said. The mathematical points or methods they planned for the lesson mostly would come from the students, though often in a different order than they anticipated.

At the beginning of the year, the teacher is the main assister but concentrates on supporting students to use all of the means of responsive assistance. This requires that students become close listeners, be collaborative and supporting, and be mutually adapting in their interactions. As the year continues, students provide a great deal of assistance in whole-class situations and increasingly in pairs or groups. All means of assistance are used to help students become better assisters, but the five Coaching means of assistance are especially important.

During Phase 1, students use modeling and instructing/explaining (always with possible assistance from the teacher) rather than these being used primarily by the teacher, as is traditional when introducing a new topic. During Phase 2 the teacher may need to do more modeling and instructing/explaining to ensure that the mathematically-desirable and accessible methods are clear to everyone, but in most classes much of this can also come from students especially after the classroom is functioning strongly. Instructing/explaining, questioning, and giving feedback (the other three Coaching means of assistance) occur most often in Phases 2 and 3 and are done by students and by the teacher. These Coaching means of assistance help students understand the learning supports introduced by the teacher and the math program for each topic and facilitate students' conscious formal learning of the formal math vocabulary, ideas, and methods.

The Vygotskiian move within the Zone of Proximal Development from other-assistance to self-assistance means that in Phase 3, all means of assistance are used less often, only on more-difficult aspects, and only for students who need them. Some students now may be observed using self-regulating speech while solving a problem; this speech may be similar to things their classmates or teacher said while solving. In Phase 4 the means of assistance may be needed very little.

Once the Nurturing Meaning-Making Math Talk Community and the seven Inquiry Learning Path Teaching means of assistance are well-established in the classroom, students can carry more of the responsibility, especially in whole-class Math Talk discussions. The teacher must always introduce the new learning supports for a new math topic, begin by eliciting student thinking, and be sure that the mathematically-desirable and accessible methods are introduced and discussed. But students later on in the year can manage much of the Math Talk. Many substitute teachers in our CMW Project classrooms commented later to the regular teacher that the students directed themselves during Math Talk and decided when they understood and were ready to practice alone. These substitute teachers initially did not even have the concept of students doing Math Talk, but the students could do it for topics they already knew without the support of the substitute teacher. We found that some students even in kindergarten are capable of high levels of assisting if they are given opportunities to do so and are helped to assist more productively.

COHERENCE AND BALANCE

Ways in which each part of the Class Learning Path Model is coherent and balanced were summarized at the beginning of the paper. Another crucial aspect of coherence and balance that is required for the Class Learning Path Model to work most effectively is programmatic coherence. This is necessary to provide adequate introduction and practice of prerequisites for a topic so that less-advanced students will be in a position to understand the mathematically-desirable and accessible methods. Effective functioning of the first three Class Learning Zone Phases for central grade-level goals requires sustained deep learning that takes time rather than a spiral approach where there is never enough time for moving everyone to mastering a mathematically-desirable and accessible method. Coherence in the learning supports across topics and across grades can reduce learning time and increase understanding and fluency, especially if these supports are chosen to allow students to experience various crucial mathematical ideas across the supports. Research to develop such coherence was a primary task of the Children's Math Worlds Research Project. The learning path curriculum that was developed in the project is now published by Houghton Mifflin as *Math Expressions.*

Of course not all topics can have an extensive period where all students explain their thinking. For some less-important topics, the teacher will go through the first three phases all in one day or in a couple of days, either because it is a small or less-important topic or because many students already have the necessary knowledge. Even in such abbreviated cases, students can still have opportunities to share their thinking and previous knowledge and practice saying any new math terms or relationships through choral practice or quick whole-class turn-taking routines. Pressures of time may even mean that teachers occasionally do much of the explaining on some days because it is faster. But it is vital that the focus on meaning-making supports is always maintained so that visual and contextual examples are always provided initially and related gesturally as well as verbally to the formal math notations and vocabulary. When mathematically-desirable and accessible methods are not available in research or in the math program a teacher is using, the current common method in the program can often be simplified to become more accessible or a student method can be used as is or adapted by the teacher or other students to become mathematically-desirable and accessible.

A final source of coherence and balance in the model is that it is helpful for subject areas other than math. Our CMW teachers often reported that Math Talk spread to an increased focus on inquiry and explanations in other subject areas. Students also spontaneously gave responsive assistance for other subject areas.

ENGLISH LANGUAGE LEARNERS

The Class Learning Path Model is effective with students from all backgrounds. But it especially simplifies the teacher's complex tasks in teaching students who must learn English as they are learning math. Garrison, Amaral, and Ponce (2006) describe their adaptation and use with teachers of Cummins' (1994) four quadrants in the LATCH model. These four quadrants are created by two axes (concrete to abstract solution strategies and context embedded to context reduced language) that result in four kinds of individual instruction teachers need to deliver within the classroom. The Class Learning Path Model simplifies this approach because all students are reached simultaneously and contribute to each other's learning. Learning for everyone initially has concrete solution strategies (e.g., Math Drawings linked to methods using formal math notation of some kind) and context embedded problems. Students higher in math skills will introduce more-advanced methods into the classroom discourse, and students higher in English will provide more-advanced explanations (that still may need to be extended by the teacher for full explanations). Thus, the English skills are modeled by classmates, and all students then need opportunities in classes to produce the relevant English words as rapid oral drills or other whole-class activities or in explanations of a solution method. As students gain experience in the topic, the problems become context reduced so as to generalize the math topic concepts. Therefore, the nice activities in the LATCH workshop outlined in the paper that have teachers sharing strategies that go into each quadrant now can go within the phases of the model: ways to create embedded context and concrete solution strategies go in Phase 1 and ways to reduce the context will be used in Phase 3 (see the LATCH Figure 2 for examples).

The effectiveness of the Class Learning Path Model in increasing English performance about math topics was indicated recently when students in a CMW school with many students identified as needing bilingual support were interviewed using the state interview of English speaking in academic areas and in everyday language. The interviewers were struck by the high levels of English students used to explain math concepts when they did not even know English words for parts of the body and other everyday English language. The teachers explained that in their Math Talk classrooms all students were expected to be able to learn to explain their thinking in English, and that with considerable modeling, they learned to do so.

CONCLUSION

Vygotsky's (1978) concept of a zone of proximal development for each student for each topic seems overwhelming to a teacher with as many as 35 students in a class (or even with "only" 20 different individuals). It suggests the need for total individualization and few whole-class activities. However, our research experience in many different classrooms for many different math topics over many years led to our simplified concept of a Class Zone of Proximal Development that operates within the four Class Learning Zone Teaching Phases to meet the needs of most students in the class by whole-class activities supported by the seven Responsive Means of Assistance within the emotional and cognitive supports of the Nurturing Meaning-Making Math Talk Community. The actual number of ways of thinking about a given situation are limited, so most or all can be discussed and examined as a way to understand the topic more deeply. The Inquiry Learning Path Teaching ensures that students are moving forward in their own learning path toward a mathematically-desirable and accessible general method. Those students who begin by knowing such a method increase their knowledge by explaining how multiple methods relate to each other and by assisting other students and the teacher within the interdependent Class Learning Zone created by the common learning supports within the Nurturing Meaning-Making Math Talk Community. Educational leaders can use the Class Learning Zone Model, with its integration of the principles from two NRC reports and from the NCTM Process Standards, to help teachers individualize their instruction to meet needs of their students within whole-class instruction.

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