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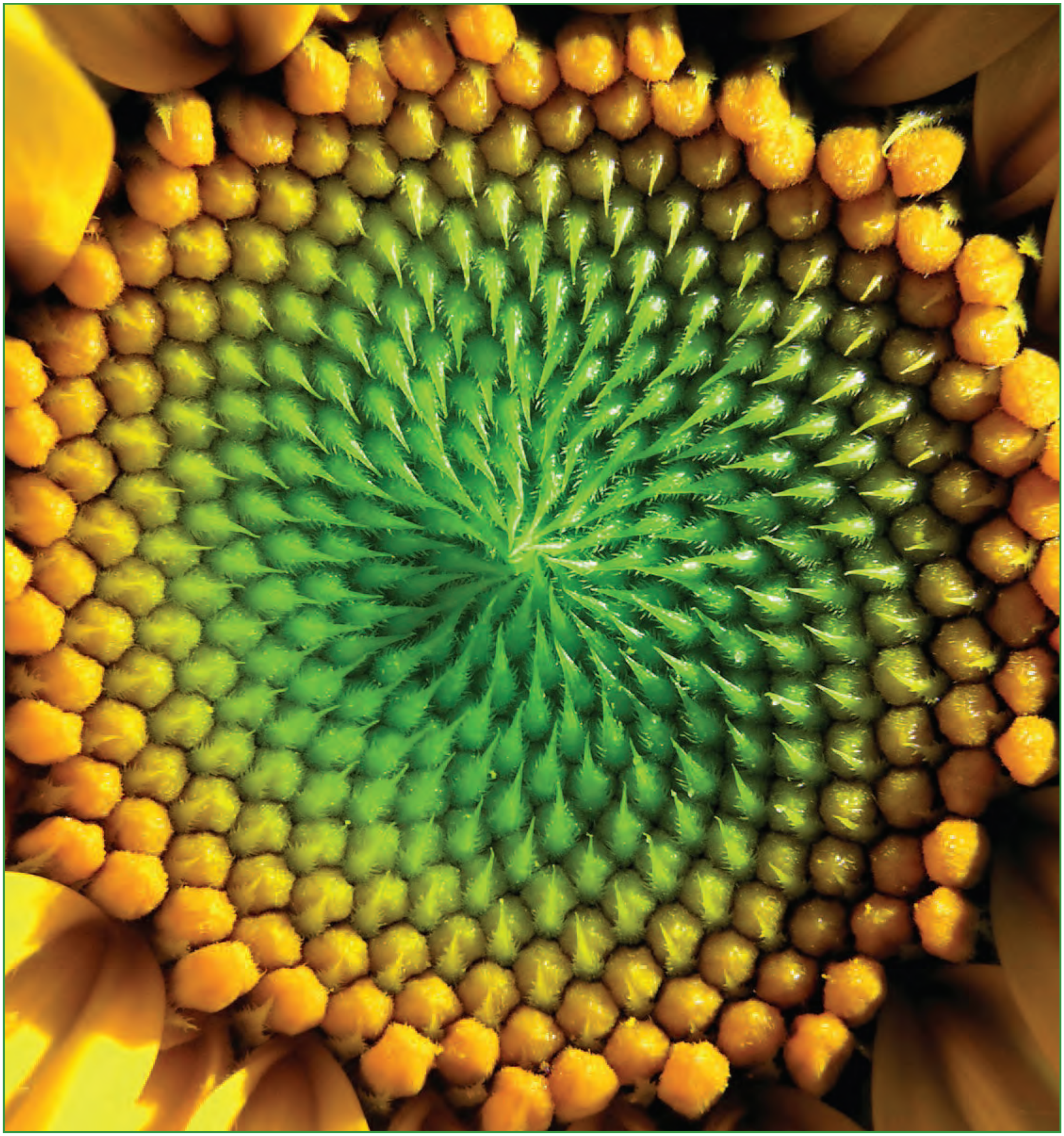

National Council of Supervisors of Mathematics www.mathedleadership.org

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Supporting Teachers' Understanding and Use of Algebra Tiles

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One of the most formidable challenges in school mathematics is helping students learn algebra. The construction of algebraic habits of mind is an important milestone in students' mathematical development (Driscoll, 1999). To reach this milestone, students must learn algebra with understanding rather than simply learning by rote (Kieran, 2007). Although conceptually oriented teaching approaches show promise for building student understanding (Senk & Thompson, 2003), making the shift to a reform-oriented curriculum is a non-trivial matter. Teachers often must re-examine their personal paradigms of mathematics instruction while also learning about new content and instructional materials. Mathematics education leaders need to be cognizant of teachers' specific professional development needs with regard to algebra teaching and learning and determine how to address them.

Learning appropriate uses for manipulatives is a common need among teachers making the shift to conceptually oriented instruction (Chval & Reys, 2008). Manipulatives afford unique opportunities for students to explore the structure of mathematical ideas, but simply placing them in front of students will not, in and of itself, improve learning (Ball, 1992; Chappell & Strutchens, 2001; Moyer, 2001). Teachers play important mediating roles in how they introduce manipulatives and encourage students to use them. Those who are most successful are able to support students without taking over the thinking involved in the problem at hand (Stein & Bovalino, 2001). In order to do this, teachers themselves must have a deep understanding of the mathematical ideas in a lesson and the manner in which manipulatives can be used to help reveal the conceptual structures associated with these ideas.

Algebra Tile Model

Algebra tile manipulatives can be used as tools to foster conceptual understanding in beginning algebra courses. A diagram of the pieces in a typical set of tiles is shown in Figure 1. For an example of the use of algebra tiles, consider the multiplication $(x + 1)(x + 3)$. The product can be interpreted as the area of a rectangle whose dimensions are $(x + 1)$ and $(x + 3)$. A diagram for performing the multiplication is shown in Figure 2. The shaded area in Figure 2 indicates that the area of a rectangle with dimensions $(x + 1)$ and $(x + 3)$ is $x^2 + x + x + x + x + 1 + 1 + 1$, or $x^2 + 4x + 3$. As another example, if students were to factor $x^2 + 4x + 3$, they could begin by building the rectangle shown in the shaded area of Figure 2. The task of factoring could then be interpreted as reading off the dimensions of a rectangle with area $x^2 + 4x + 3$. Doing so

FIGURE 1. *Pieces in a typical set of algebra tiles*

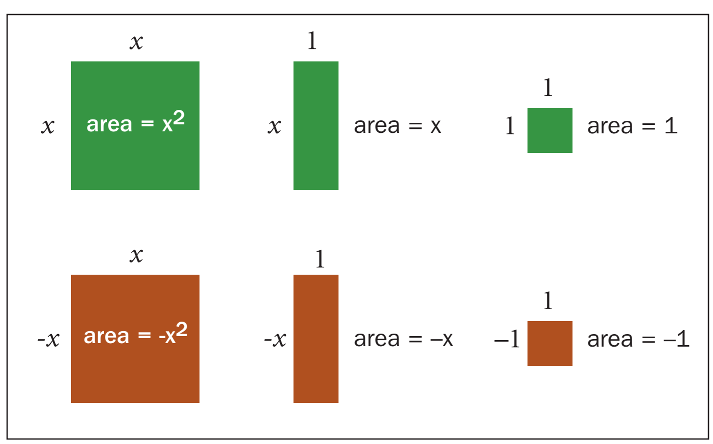
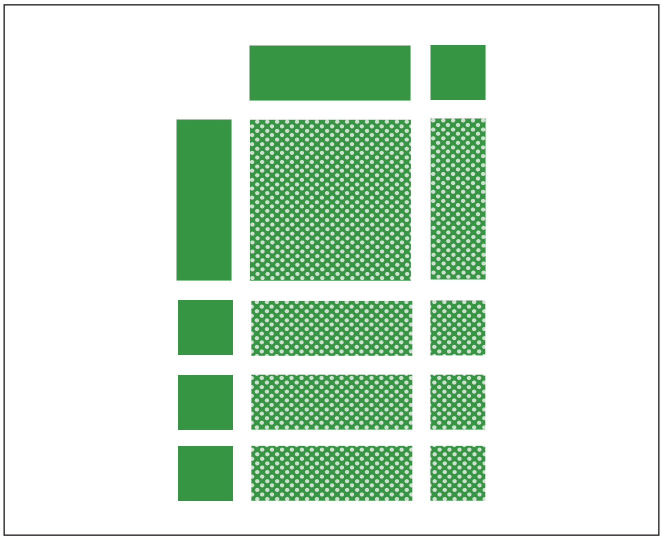


FIGURE 2. *Algebra tile diagram for polynomial multiplication and factorizations*



would give the factorization $(x + 1)(x + 3)$. Additionally, one could interpret the $x^2 + 4x + 3$ area as a dividend. The $x + 1$ length could then be thought of as a divisor, with the width, $x + 3$, being the quotient determined from constructing a rectangle with and area of $x^2 + 4x + 3$ and length of $x + 1$.

Lesson Study Professional Development

We had the opportunity to work with a group of four high school mathematics teachers as they used algebra tiles with their students as part of a university-sponsored lesson study project. The group was established as part of a larger funded project that included three school districts (Groth, 2011). The teachers were new to the lesson study process and to using algebra tiles with students. One teacher in the group, Janet (all teacher names are pseudonyms), had recently learned of algebra tiles at a professional conference. When presented the opportunity to engage in lesson study, Janet suggested to the group that they focus on becoming better acquainted with the tiles and using them to teach polynomial factoring. The other teachers in the group agreed that polynomial factoring was a difficult topic for their students and were willing to explore the potential of the algebra tiles to teach polynomial factoring during their lesson study work together. Because the algebra tile model was new to them, they expressed interest in the opportunity to learn how the algebra tile model might be used to address the mathematics content they were responsible for teaching by working with their colleagues and university faculty during lesson study.

The lesson study process for the project is depicted in Figure 3. Its structure allowed teachers to gradually polish and refine ideas for instruction as they worked together. The rectangles in Figure 3 represent the phases in a lesson study cycle. Arrows between the rectangles indicate the progression that occurred from one phase to the next. Teachers were given one semester to progress through each cycle and completed two cycles (see Table 1). As indicated in Figure 3 and in Table 1, the first phase consisted of constructing a lesson collaboratively. The lesson study goals were not dictated by university personnel. Instead, teachers chose learning goals in collaboration with one another (Lewis & Tsuchida, 1998). Once the goal of teaching

FIGURE 3. *A lesson study professional development model*

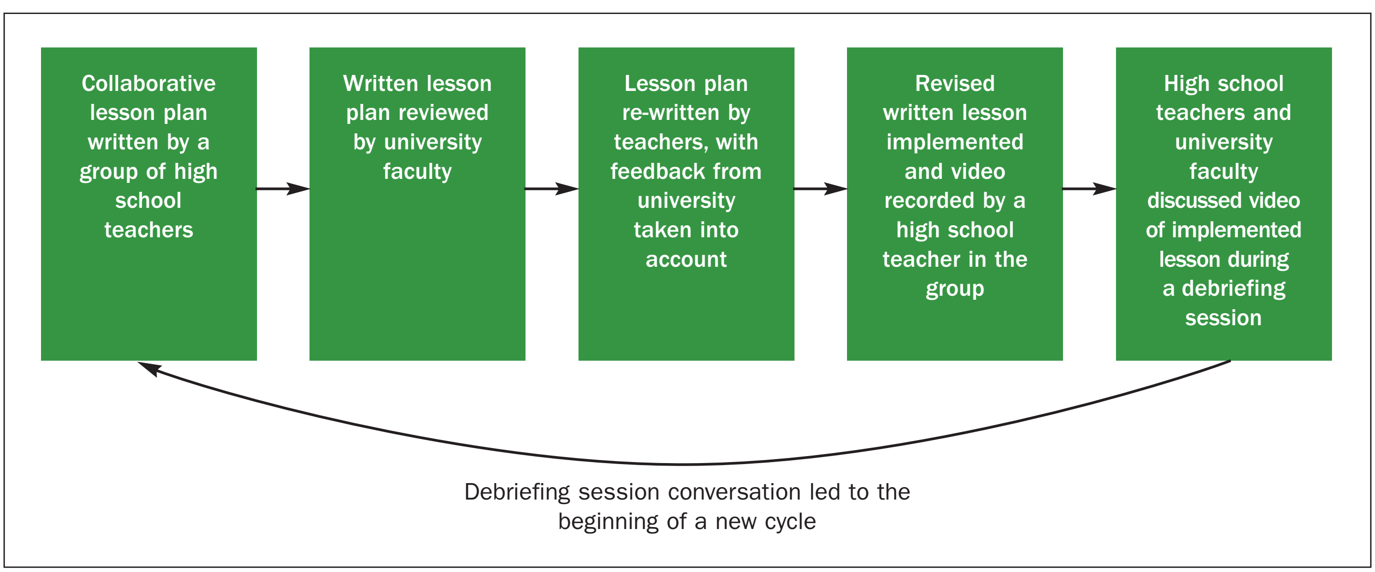


Table 1 — Lesson Study Timeline

Fall semester: Lesson study cycle 1 **Lesson-implementing teacher: Janet**	**Spring semester: Lesson study cycle 2** **Lesson-implementing teacher: Martha**
1F. Algebra tile lesson designed collaboratively by teachers 2F. Algebra tile lesson reviewed by university faculty 3F. Teachers re-wrote lesson, taking university faculty feedback into account 4F. Janet implemented the lesson with her class and it was video-recorded 5F. Debriefing session occurred in which university faculty and teachers viewed and discussed video-recorded lesson. This motivated another cycle of lesson study in the spring dedicated to improving algebra tile usage during instruction.	1S. Algebra tile lesson from the fall cycle revised by teachers using feedback from cycle 1 2S. Revised written lesson reviewed by university faculty 3S. Teachers re-wrote lesson, taking university faculty feedback into account 4S. Martha implemented the lesson with her class and was video-recorded 5S. Debriefing session occurred in which university faculty and teachers viewed and discussed video-recorded lesson. Discussion focused on how the lesson could be further refined and improved.

polynomial factorization using algebra tiles was identified, the teachers collaboratively wrote a lesson to be implemented by one of the members of the group.

After the teachers wrote their lesson plan, it was sent to university faculty for review. Two university mathematics faculty members reviewed each lesson. The faculty reviews were solicited because previous research has shown that perspectives from outside a lesson study group can be valuable for identifying areas for improvement in lessons (Fernandez, 2005; Yoshida, 2008). Reviewers made written comments on potential strengths and weaknesses of the lessons. Within two weeks of the submission of each lesson, university faculty feedback was sent to the teachers. Teachers were then asked to use the feedback, to the extent they judged feasible, to revise the lesson before implementing it. Throughout the process, they had the freedom to accept or reject any feedback using their professional judgment.

After the initial written lesson was revised, one member of the lesson study group taught it while another member of the lesson study group video recorded it. The videos were later viewed by all of the teachers during a debriefing session, along with the second and third authors of this article. Although it is ideal to have all of the teachers present in the room when the lesson is implemented (Lewis, 2002), when schedules do not allow for this, a video of the lesson can be used as an alternative. During each debriefing session (one per cycle), the teacher who implemented the lesson and the videographer provided context to help explain the lesson video. After this information was shared, the video was played, and debriefing session participants took notes on perceived strengths and weaknesses of the lesson. When the video concluded, each individual participating in the meeting was prompted to share his or her perceptions. The arrow from phase 5 (debriefing) to phase 1 (planning) in Figure 3 shows that the debriefing session conversation from the first lesson study cycle sparked a second cycle.

As the lesson study process was carried out, several artifacts were retained to record its history and analyze teachers' learning:

- Written lessons produced by the teachers;
- Written feedback on the lessons given by university faculty;
- Video recordings and transcripts of the lessons teachers implemented during the first and second cycles of the lesson study; and
- Audio recordings and transcripts of the debriefing session conversations involving the university faculty and teachers during the first and second cycles of the lesson study.

The authors of this article collaboratively analyzed the artifacts to identify key learning experiences during the project. Our goal in doing so was to help other mathematics education leaders anticipate elements of teachers' knowledge that may need development and support as they begin to use algebra tiles and other manipulatives to address important mathematics content.

Key Learning Experiences for Teachers Beginning to use Algebra Tiles

In the remainder of this article, we describe what we consider to be the most important elements of conversations that occurred as teachers learned to use algebra tiles:

- Grounding students' work with algebra tiles in the concept of area;
- Helping students understand how the length of a tile is meant to represent a variable quantity;
- Using algebra tiles to establish the conceptual ground for factoring rather than just illustrating procedures;
- Choosing polynomials with the potential to encourage genuine problem-solving; and
- Encouraging problem-solving classroom discourse with algebra tiles.

We conjecture that several of these elements apply not only to the group of teachers described in this article, but will also apply to other teachers learning to use algebra tiles. Some of the elements also apply to the use of manipulative models in general, such as focusing on concepts, problem solving, and rich classroom discourse.

GROUNDING STUDENTS' WORK WITH ALGEBRA TILES IN THE CONCEPT OF AREA

Although the algebra tile model is based upon dimensions and areas of rectangles, it was challenging for teachers to think about how to make this connection in their classrooms. In the group's first collaboratively written lesson, the relationship between algebra tiles and area was not mentioned at all. When the teachers sent the written lesson to us for review, we recommended that they connect students' previous experiences with area outside of algebra classes to work with polynomials. One recommended activity was to have students make as many arrangements of 12 unit blocks as possible to find the factors of 12 before doing similar work with trinomials. This work would then lead to situations where length and width were variable quantities.

After the reviewers' comments were shared with the lesson study group, the teachers met with each other to decide how to use the feedback to edit the initial written lesson. Janet (a pseudonym, as are the rest of the teacher names in this report) then taught the revised lesson in her class. One striking feature of the lesson video was that the word "area" was not used at all in reference to the tiles. Janet gave a name to each piece in the algebra tile set, and then characterized the task of factoring a polynomial as arranging the appropriate pieces into a rectangle. This made each factorization into a jigsaw puzzle-like task to perform. When Janet wanted to prompt students to provide the factors of a given polynomial, she asked the question "What do these have in common?," being somewhat ambiguous about what "these" referred to. When students did not respond to the initial question, she would ask a more directive question such as "How many columns are there?" so a student would offer the correct response. If the idea of area had been used in connection with the tiles, it would have allowed her to focus students' attention instead on the question, "What are the dimensions of a rectangle with the area of the given polynomial?" During the debriefing session for this lesson, we again mentioned the connection to area and encouraged teachers to make this connection during the cycle 2 lesson.

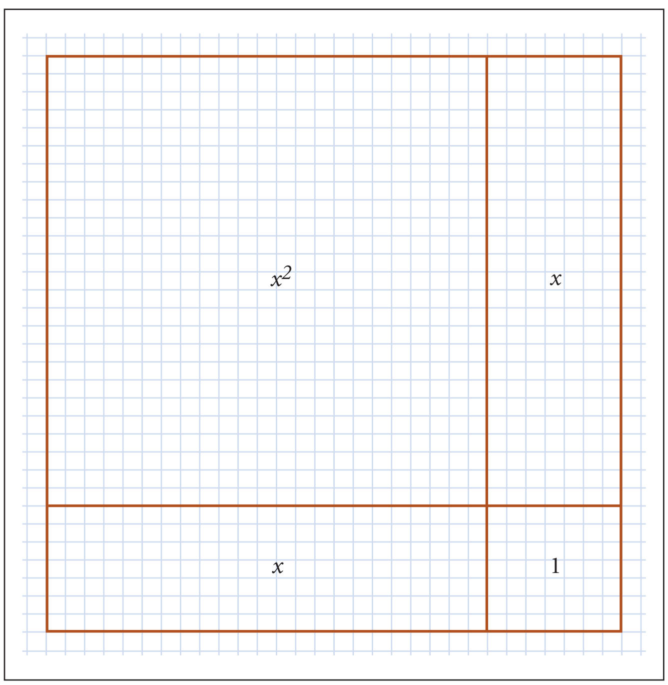
When implementing the cycle 2 lesson, the teachers *did* attempt to connect the concept of area to the algebra tile model. The implementing teacher for the cycle 2 lesson, Martha, asked students to think about what a 3 by 3 rectangle would look like with the unit tiles. She then asked students to think about an x by x rectangle, encouraging them to generalize the model to rectangles with variable lengths. Despite this progress from the cycle 1 lesson, it still proved difficult to connect the algebraic concepts represented by area to the lesson at points where this connection would be useful. For example, after Martha began discussing a rectangle with dimensions $(x + 2)$ and $(x - 3)$ with students, the word "area" was not used for the remainder of the lesson. Instead, when students were given factoring problems, Martha told them to decide what was "above" and "to the left" of the "box" rather than to find the dimensions of a rectangle with a given area. During the debriefing session for this lesson, we once again took the opportunity to suggest that teachers frame polynomial factorization tasks as determining the length and width of a rectangle whose area is represented by a given polynomial. After viewing the cycle 2 lesson video, teachers recognized this as being necessary for strengthening student's conceptual understanding.

HELPING STUDENTS UNDERSTAND HOW THE LENGTH OF A TILE IS MEANT TO REPRESENT A VARIABLE QUANTITY

Another key instructional decision discussed during debriefing sessions was that teachers required students to put their algebra tile diagrams for factorization problems on grid paper. This occurred during both the cycle 1 and cycle 2 lessons. Although this move produced neat-looking student papers, it also implied that the x-length in the model had a set integer value. (See Figure 4.) Algebra tile manipulatives are intentionally constructed so that unit tiles cannot precisely measure out the x-length, but when drawing tile pieces on graph paper, this structural characteristic of the model is lost. During debriefing sessions, teachers initially supported the decision to use grid paper for algebra tile diagrams because they felt the grid paper and colored pencils helped students organize their work. Janet, however, did express concern that having students work on grid paper caused some to think that the pieces representing variable length actually represented a fixed length of a certain number of grid squares. She acknowledged hearing students express this misinterpretation of the model as she circulated about the classroom during the cycle 1 lesson. Janet was satisfied that the problem was addressed, however, through her individual conversations with students as they worked.

Teacher leaders guiding teachers' first experiences teaching with algebra tiles would do well to keep in mind the problematic nature of representing variable lengths with fixed plastic pieces. Designers of algebra tiles try to address this dilemma by making unit squares that will not divide

FIGURE 4. *Grid paper sketch of an algebra tile diagram*



lengths that represent variable quantities. This technique, however, has the potential to cause almost as much confusion as having students sketch tiles on grid paper. While grid paper sketches imply that the variable quantities have fixed integer values, the tiles themselves can be interpreted to represent variable quantities as fixed non-integer values. To overcome this conceptual hurdle, it can be helpful to consider online virtual manipulatives. Some online versions of algebra tiles (e.g., http://nlvm.usu.edu/en/nav/vlibrary.html) (Cannon, Dorward, Heal, & Edwards, 2001) use built-in sliders that allow users to change the length of x to portray variable quantities. As the sliders are moved, the side lengths in algebra tiles and the rest of the diagram change accordingly. Once students understand the fundamental idea of variable side lengths, their work with plastic tiles or grid paper can be recognized as portraying just one possible value for x.

USING ALGEBRA TILES TO ESTABLISH THE CONCEPTUAL GROUND FOR FACTORING RATHER THAN JUST ILLUSTRATING PROCEDURES

The algebra tile model is meant to make polynomial multiplication and factorization accessible to students by building a bridge from students' understanding of area to the development of more formal algebraic techniques. In some cases during the lesson study project, this intended sequence was essentially carried out in reverse. That is, students were asked to use the tiles to check or illustrate the result of using previously learned procedures for polynomial factorization and multiplication. Part of this was due to the fact that students had studied conventional techniques for polynomial factorization and multiplication earlier in the year, and rather than attempting to ignore previous instruction on the topic, the group built it into their plans.

At the outset of the project, the lesson study group wrote in their lesson plan that they considered knowledge of polynomial multiplication to be a necessary prerequisite for working with algebra tiles. At some points in the first implemented lesson, Janet asked students to check their work by multiplying polynomials symbolically. This move encouraged students to appeal to a procedure to judge the correctness of their answers rather than using the structural characteristics of the tile model to understand and justify the formal procedure. The tiles then essentially became a way to illustrate a procedure rather than a means to provide conceptual grounding for the process of factoring polynomials. This difficulty lingered through the second implemented lesson. In that lesson, Martha at one point told students, "You will be using your knowledge of factoring to figure out what goes in the box." When a student presented an incorrect algebra tile diagram for factoring $x^2 - 2x + 1$ to the rest of the class, Martha explained why it was incorrect by demonstrating a conventional symbolic procedure for factoring she had previously taught. She *did*, however, seem to believe that there were limitations in appealing to previously-learned procedures to understand a concept. For instance, she told students at the outset of the lesson that it would have been ideal to use the tiles earlier in the year, when they were just beginning to learn polynomial multiplication and factoring.

Failure to ground students' work with algebra tiles strongly in the idea of area appears to be a cause of having to appeal to procedures to determine the correctness of student work. When the concept of area is missing from an algebra tile lesson, the primary means for determining the correctness of any given factorization is to check (or create) tile arrangements by going back to previously-learned procedures. Therefore, during the debriefing session for the second implemented lesson, university faculty again recommended making a stronger connection between the algebra tiles and the idea of area. It was suggested that teachers frame polynomial factorization tasks as determining the length and width of a rectangle whose area is represented by a given polynomial. Doing so allows students to check their work by examining the dimensions and area of rectangles instead of relying solely upon memorized procedures. The teachers took up discussion of this idea during the debriefing session for the second lesson they implemented, and began to consider using the algebra tiles at the outset of instruction instead of waiting until students had learned procedures for polynomial multiplication and factoring. Introducing the tiles before formal procedures gives students a chance to recognize formal symbolic procedures as convenient abbreviations of their concrete work with the tiles. Teachers did not have the opportunity to pursue this sort of re-sequencing of instruction during lesson study, but the lesson study process introduced the idea as a goal for future work.

CHOOSING POLYNOMIALS WITH THE POTENTIAL TO ENCOURAGE GENUINE PROBLEM-SOLVING

Teachers beginning to use algebra tiles also should develop the art of careful task selection. In the first written lesson teachers produced during the lesson study project, the tasks they selected largely resembled those that would be given using a conventional approach to teaching factorization. In some cases, these tasks happened to be suitable for tile-based approaches (e.g., factor $x^2 + 6x + 9$). In other cases, the tasks used time inefficiently because of the large number of tiles that would be necessary for producing the accompanying representations (e.g., factor $x^2 + 15x + 36$). When we reviewed the first written lesson, we suggested changes in the included tasks students were to perform. Polynomials with negative coefficients were missing from the problem set, so their inclusion was recommended in place of polynomials that simply required a large number of tiles. Teachers incorporated this suggestion into the implemented lesson during the in-class tasks students were to perform and as part of the homework assigned for the day.

Despite teachers' acceptance of suggestions on task alteration during the first cycle of lesson study, their second written lesson indicated that it would be profitable for them to continue to delve more deeply into the issue of task selection. In the written lesson plan the teachers produced at the beginning of cycle 2, they noted that "some students will have difficulty as the numbers get bigger," showing that they still considered the absolute values of the coefficients to be the primary determinant of problem difficulty. In a review of the second lesson, it was noted that the constant term in each polynomial the teachers planned to present was positive, so including some with negative constant terms for students ready for such a challenge was suggested. Non-prime constant terms were suggested as another means of increasing the level of challenge, since the number of rectangular arrays that can be formed is greater than for primes. It was also suggested that non-factorable trinomials be included among the problem set in order to cause student discussion of the characteristics of such polynomials.

Some of the written suggestions from the lesson reviews were incorporated into the tasks teachers used in the implemented lesson although the changes were mostly to smaller features of individual tasks. For instance, the suggestion to include polynomials with negative constant terms was adopted by asking students to factor $x^2 - x - 6$. However, the idea of including some polynomials that would not factor was not implemented. Instead, at one point during the cycle 2 lesson, Martha stated, "You will see that it will work out, and if it doesn't work out, you mess around until you get one (a rectangle)." Some of her students may have interpreted this statement to mean that any quadratic could be factored with algebra tiles even though Martha may have only intended to convey that all of the polynomials given in class and in the text could be factored with the tiles. Hence, although teachers' attention was drawn to some of the subtle but important differences in tasks suitable for algebra tiles during lesson study, task selection persisted throughout the project as an important area for further attention.

When working with teachers on framing tasks, teacher leaders may find it useful to incorporate the levels of cognitive demand framework described by Smith and Stein (1998). It provides a means for explicitly discussing the types of student thinking required in tasks posed by teachers. The four levels in the framework can be summarized as follows:

- ***Memorization:*** tasks that simply require memorization of facts, rules, or definitions;
- ***Procedures without connections:*** tasks whose completion relies solely upon execution of previously learned procedures;
- ***Procedures with connections:*** tasks that require students to use procedures but also prompt them to explore the procedure's conceptual underpinnings; and
- ***Doing mathematics:*** tasks that have no prescribed solution method; students need to draw on conceptual knowledge to devise solution strategies.

Although we prompted the group to reach toward higher levels of demand by suggesting tasks that required novel student thinking (e.g., non-factorable polynomials and those with negative coefficients), we did not explicitly share the four-level framework. In subsequent professional development work, the first two authors have found the framework to be useful for fostering meaningful conversations about higher-level tasks. Similarly, Arbaugh and Brown (2005) found that using the framework helped improve teachers' choice of tasks. Such experiences suggest that it may be profitable for lesson study facilitators to explicitly introduce the notion of level of cognitive demand as a device to help teachers choose and design genuine problem-solving tasks.

ENCOURAGING PROBLEM-SOLVING CLASSROOM DISCOURSE WITH ALGEBRA TILES

A final observation addresses the contrast between the classroom discourse in the lessons implemented during cycles 1 and 2. In posing the first factoring problem in the lesson, Janet started with the premise that the tiles comprising the polynomial must form a rectangle. Rather than showing students precisely how to form the rectangle, she took suggestions from the class. As the class worked, she asked some students to demonstrate their strategies for forming rectangles for different polynomials. At times, this meant that trial-and-error strategies were demonstrated by students. On the other hand, during the cycle 2 lesson, Martha tended to funnel students toward the solution she was looking for by asking a series of questions that required only one-word or one-number responses.

Part of the reason for the difference between the two classrooms may have been that the scaffolding questions to be asked as the lesson moved from example to example were not specified in the group's written lesson. This allowed for a greater degree of individual interpretation about which types of questions would be most effective. Therefore, as teachers construct written lessons, it can be useful for supervisors to work with them to decide how questions will be posed before implementing the lesson. Although it would be nearly impossible to write all of the questions teachers are to ask during a lesson, it is helpful to have consensus that the types of questions posed will encourage students' intellectual engagement with problem solving rather than restricting their thinking with narrow questions and directives.

As teacher leaders work with teachers to help encourage problem-solving discourse, the NCTM (2000) communication process standard can be a useful reflective device. It emphasizes the importance of student-to-student communication, stating that students should "analyze and evaluate the mathematical thinking and strategies of others" (p. 60). It also connects classroom discourse to task selection, stating, "Students need to work with mathematical tasks that are worthwhile topics of discussion. Procedural tasks for which students are expected to have well-developed algorithmic approaches are usually not good candidates for such discourse" (p. 60). As teachers view lesson video during debriefing sessions, they can be encouraged to analyze the extent to which the classroom discourse models the recommendations of the process standard. In cases where alignment is lacking, the process standard can help in the diagnosis of root causes. For instance, in the lesson Martha implemented, opportunities for student-to-student communication were lacking. She also tended to lower the levels of cognitive demands of tasks by providing many directive questions while teaching. Tracing problematic aspects of classroom discourse back to their root causes provides information that can be used to re-structure and polish instructional plans during lesson study.

Conclusion

We hope this paper offers insights to mathematics instructional leaders about challenges they may encounter as they help teachers begin to use algebra tiles with their students. We have described a number of ideas teachers learned as they engaged in collaborative planning and conversation, but it should be noted that we gained just as much from the opportunity to interact with the group. As we reflected on the group's progress and challenges, we began to re-think the ways in which we introduce algebra tiles in our university level mathematics and mathematics education courses for pre-service and in-service teachers. In particular, we now focus more strongly on the connection to area as the underpinning conceptual ground for the algebra tile model. Once that connection is established, it becomes easier to discuss the ideas of tile length as a variable quantity, the distinction between illustrating procedures and teaching concepts, the importance of choosing appropriate polynomials for students' work, and optimal classroom discourse patterns for tile use. As teachers learn to construct lessons incorporating these elements, their students gain rich opportunities to learn algebra with understanding.

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