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Standard Algorithms in the Common Core State Standards

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In a recent issue of the *Journal for Mathematics Education Leadership* comparing many state standards to the Common Core State Standards for Mathematics (CCSS-M), Reys and Thomas (2011) noted the following

> "[The] specific statement of the culminating standard for each operation in CCSS-M includes the expectation of use of 'the standard algorithm.' … However, a definition for 'the standard algorithm' is not offered. If the authors of CCSS-M had a particular standard algorithm in mind, it was not made explicit nor is an argument offered for why a particular (standard) algorithm is expected." (p.26)

The issue of standard algorithms was addressed in the Number and Operations in Base Ten (NBT) Progression document written by individuals involved in the creation of the CCSS-M as a narrative discussion of the learning progression of standards within a particular domain across grade levels. This progression document can be found at http://commoncoretools.me/category/progressions/. Questions about the standards and the learning progressions associated with them have also been discussed by authors of the CCSS-M at the http://commoncoretools.me web site. This article draws on the CCSS-M, the NBT Progression document, and the webpage dialogue with the authors of the CCSS to explore the question of what is meant by a standard algorithm.

For multidigit computation, the CCSS-M specifies a learning progression in which students develop, discuss, and use efficient, accurate, and generalizable methods based on place value and properties of operations. Students explain the reasoning used in a written method with visual models. Then, in a later grade, students move to using the standard algorithm fluently with no visual models. While the CCSS-M includes specific standards addressing the understanding of place value, in this article we focus only on the standards addressing multidigit computation, though it is vital to understand that an important goal of these computation standards is to deepen and extend place value concepts and skills.

The National Research Council report *Adding It Up* (Kilpatrick, et al., 2001) described many variations of algorithms that are used in the United States, and there have been analyses and discussions about which variations of these algorithms might be best used for at least the last century. Variations of algorithms also exist in other countries. For instance, Fuson and Li (2009) identified a number of variations of algorithms for multidigit addition and subtraction found in textbooks in China, Japan, and Korea. It is important to ask what is intended by the term *the x* of the CCSS-M chose to use the term *the standard algorithm* rather than *a standard algorithm*.

In a dialogue posted on April 25, 2011, CCSS-M lead author Bill McCallum suggested that the following explanation from the NBT Progression document would help address the question of what is meant by *the standard algorithm* or *a standard algorithm:*

> "In mathematics, an algorithm is defined by its steps and not by the way those steps are recorded in writing. With this in mind, minor variations in methods of recording standard algorithms are acceptable." (NBT, p13)

The NBT Progression document also defines a standard algorithm as follows:

> "Standard algorithms for base-ten computations with the four operations rely on decomposing numbers written in base-ten notation into base-ten units. The properties of operations then allow any multi-digit computation to be reduced to a collection of single-digit computations. These single-digit computations sometimes require the composition or decomposition of a base-ten unit." (p13)

Taken together, the NBT Progression document summarizes that *the standard algorithm* for an operation implements the following mathematical approach with minor variations in how the algorithm is written:

- Decomposing numbers into base-ten units and then carrying out single-digit computations with those units using the place values to direct the place value of the resulting number; and
- Using the one-to-ten uniformity of the base ten structure of the number system to generalize to large whole numbers and to decimals.

In the remaining portions of this article we identify variations in written methods for recording the standard algorithm for each operation, discuss what we believe could be considered minor variations in these algorithms, and suggest criteria for evaluating which variations might be used productively in classrooms. We also clarify the terms strategy, written method, and standard algorithm—all of which are used in the CCSS-M and the NBT Progression document —and suggest ways in which leaders in mathematics education can use the information in this article to help teachers and students understand and become fluent in base-ten computation.

Strategy, Standard Algorithm, and Written Method

The key NBT standards and related excerpts from the grade level introductions to the critical areas from the CCSS-M are given for multidigit addition and subtraction in Table 1 and for multidigit multiplication and division and all operations on decimals in Table 2. In all grades including Grade 1 students are to develop, discuss, and use efficient, accurate, and generalizable methods. The initial methods use strategies based on place value and properties of operations; these are related to written methods and the reasoning is explained using visual models (concrete models or drawings in Grades 1 and 2 and drawings/diagrams in Grades 4 and 5).

The word "strategy" emphasizes that computation is being approached thoughtfully with an emphasis on student sense-making. *Computation strategy* as defined in the Glossary for the CCSS-M includes special strategies chosen for specific problems, so a strategy does not have to generalize. But the emphasis at every grade level within all of the computation standards is on efficient and generalizable methods.

For each operation, as discussed above, there is a particular mathematical approach that is based on place value and properties of operations; an implementation of the particular mathematical approach is called the *standard algorithm* for that operation. To implement a standard algorithm one uses a systematic *written method* for recording the steps of the algorithm. There are variations in these written methods. Some of these variations are a little longer because they include steps or math drawings that help students make sense of and keep track of the underlying reasoning. Over time, these longer written methods can be abbreviated into shorter written methods that allow students to achieve fluency with the standard algorithm while still being able to understand and explain the method.

We have discussed above that standard algorithms rely on the particular mathematical approach of decomposing numbers into base-ten units and then carrying out single-digit computations with those units. They are efficient because they use place-value knowledge and single-digit computations that have already been developed. Because of the consistent one-for-ten structure across all whole number and decimal places, these algorithms thus generalize to large whole numbers and to decimals. As Bill McCallum says (April 29, 2011) about Grade 2: "Using three digits rather than two allows one to illustrate the iterative nature of the algorithms, and emphasize the fact that the base ten system uses the same factor, 10, for each rebundling of units into higher units." The standard algorithms are especially powerful because they make essential use of the uniformity of the base-ten structure. This results in a set of iterative steps that allow the algorithm to be used for larger numbers. For addition and subtraction, this is first visible for totals larger than 100.

Table 1. NBT Standards that Focus on Multidigit Addition and Subtraction and Related Grade-Level Critical Areas

GRADE 1 CRITICAL AREA (2) Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones.

Grade 1: Use place value understanding and properties of operations to add and subtract.

4. Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

GRADE 2 CRITICAL AREA (2) Students use their understanding of addition to develop fluency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations.

Grade 2: Use place value understanding and properties of operations to add and subtract.

5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

9. Explain why addition and subtraction strategies work, using place value and the properties of operations. [Explanations may be supported by drawings or objects.]

GRADE 3 CRITICAL AREA There is no critical area for multidigit computation.

Grade 3: Use place value understanding and properties of operations to perform multi-digit arithmetic.

2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

GRADE 4 CRITICAL AREA (1) Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. (This continues in Table 2.)

Grade 4: Use place value understanding and properties of operations to perform multi-digit arithmetic.

4. Fluently add and subtract multi-digit whole numbers using the standard algorithm.
[Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.]

Criteria for Emphasized Written Methods

In the past, there has been an unfortunate dichotomy suggesting that *strategy* implies understanding and *algorithm* implies no visual models, no explaining, and no understanding. In the past, teaching *the standard algorithm* has too often meant teaching numerical steps rotely and having students memorize the steps rather than understand and explain them. The CCSS-M clearly do not mean for this to happen, and the NBT Progression document clarifies this by showing visual models and explanations for various written methods for standard algorithms for all operations. General methods that will generalize to and become standard algorithms can and should be developed, discussed, and explained initially using a visual model. Given this emphasis on meaning-making, variations in ways to record the standard algorithm that support and use place value correctly should be emphasized. Given the centrality of single-digit computations in algorithms, variations that make such single-digit computations easier should be emphasized. Different written methods for

Table 2. NBT Standards that Focus on Multidigit Multiplication and Division and Related Grade-Level Critical Areas and on All Operations with Decimals

GRADE 4 CRITICAL AREA (1) Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop fluency with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate and mentally calculate quotients, and interpret remainders based upon the context.

Grade 4: Use place value understanding and properties of operations to perform multi-digit arithmetic.

5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

6. Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

GRADE 5 CRITICAL AREA (2) Students develop understanding of why division procedures work based on the meaning of base-ten numerals and properties of operations. They finalize fluency with multi-digit [addition, subtraction] multiplication and division. They apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They [develop fluency in these computations, and] make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of decimals to hundredths efficiently and accurately.

Grade 5: Perform operations with multi-digit whole numbers and with decimals to hundredths.

5. Fluently multiply multi-digit whole numbers using the standard algorithm.

6. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction (and between multiplication and division); relate the strategy to a written method and explain the reasoning used.

GRADE 6 CRITICAL AREA There is no critical area for multidigit computation.

Grade 6: Compute fluently with multi-digit numbers and find common factors and multiples.

2. Fluently divide multi-digit numbers using the standard algorithm.

3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Note: There are two glitches in the Critical Area for Grade 5. The words in brackets are not consistent with the standards themselves, so should be omitted. In 5.NBT.7 the words "and between multiplication and division" should follow "between addition and subtraction" and so were inserted there in parentheses. Also, the word *procedures* is used beginning in Grade 4 rather than the word *methods* used in earlier grades. No change in meaning is intended (e.g., Grade 4 "procedures" are not more rote than are Grade 2 "methods").

recording standard algorithms vary in three additional features. First, they always involve different kinds of steps, e.g., ungrouping (borrowing) to be able to subtract and the actual subtracting. These kinds of steps can alternate or can be completed all at once. Variations in which the kinds of steps alternate can introduce errors and be more difficult. Second, variations can keep the initial multidigit numbers unchanged, or single-digit numbers can be written so as to change (or seem to change) the original numbers. The former variations are conceptually clearer. Third, many students prefer to calculate from left to right, consistent with how they read numbers and words, so variations that can be undertaken left to right are helpful to many students.

Multidigit Addition

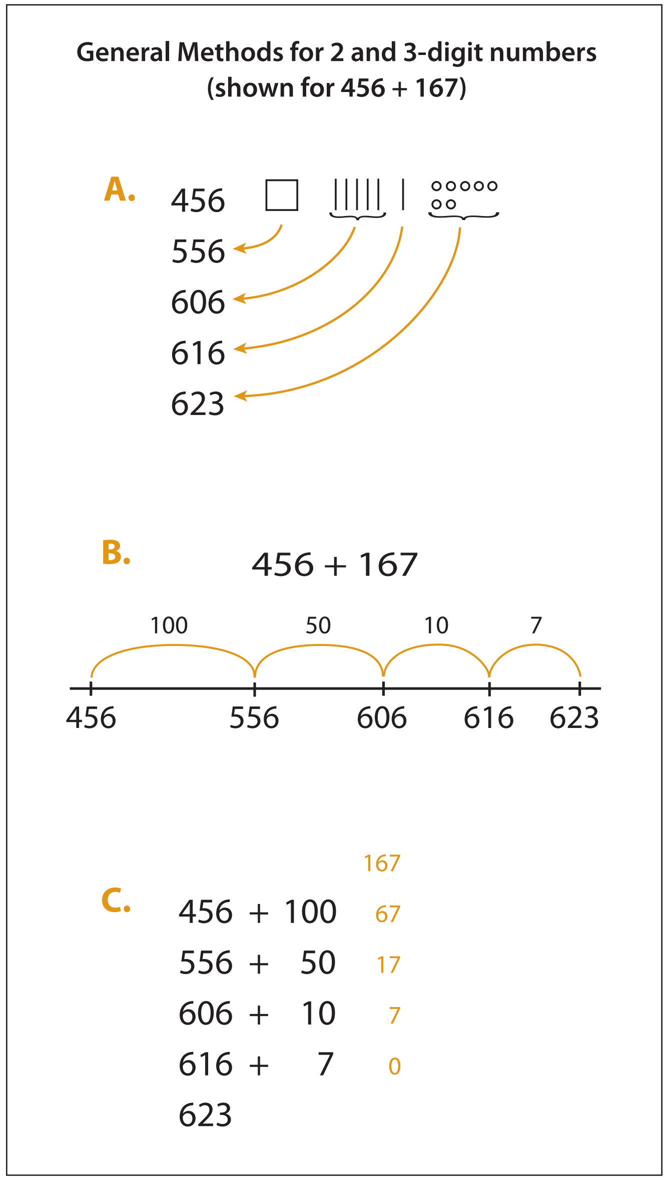
Now let us examine the examples of written methods for the standard algorithm given in the NBT Progression document and some other common strategies and consider the issues of how much variation in a written method is sensible and which variations might be emphasized. In the research literature, two approaches to multidigit addition and subtraction have been identified. (See Fuson, 1990, 1992; Fuson, et al., 1997; Verschaffel, et al., 2007):

a. decomposing into base-ten units (also called collection-based or split), which is the approach of the standard algorithm and

b. beginning with one undecomposed number (also called sequence or jump).

Figure 1 shows a count or add on approach that begins with one undecomposed number. With written methods for such an approach, one needs to keep track at each step of how much of the second addend one has already counted or added on. Two of the several variations of such keeping track methods using visual models are shown as Methods A and B. Variations of what is counted or added on first are possible (e.g., one might add on 4 to make 456 be 460), and the number of steps involved can vary. Method C keeps track numerically rather than with a drawing. Other variations are shown and discussed in Fuson, et al. (1997) and in Verschaffel, et al. (2007) and in NCTM (2010, 2011). These written methods are general methods for all 3-digit numbers but they are not practical for larger numbers. Even with these 3-digit numbers, one can see that it is a bit tricky to keep track of which places in the increasing total change at each step and perhaps even to notice explicitly that one is adding on like units. These methods are easier for 2-digit numbers and may arise

FIGURE 1. *Multidigit Addition Methods that Begin with One Undecomposed Number (Count or Add On)*



when students extend counting on methods with single-digit numbers.

Written methods for the standard algorithm that are generalizable to larger numbers and to decimals use single-digit computations of place-value units and are given in Figure 2. A drawing that could be used to direct or make sense of any of these written methods is given in the top row. These place-value drawings can help direct the steps in these methods by stimulating adding like units and composing as needed, as specified in the CCSS-M: *one adds hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose tens or hundreds.*

Writing the addends above each other helps students add like units for the methods that decompose base-ten units. Students can begin to use any of these methods without drawings whenever they no longer need these visual models, though they might make such drawings when explaining their method to classmates who might still need such a sense-making support. Dropping the drawings could begin to happen at Grade 1 or 2, but it definitely should be happening at Grade 3 so that students can focus on extending their method to larger numbers in Grade 4. Drawings could be used initially in Grade 4 especially for thousands, but they do not have to be used.

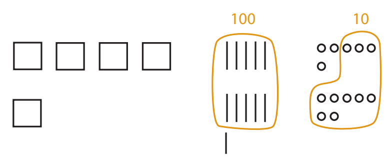
The expanded notation Method D that showed the sum of each place value can be extended to 1,000,000 but it could be difficult to keep all of the place value columns straight. This is an example of a written method that shows extra steps (helping steps) that can be very useful when students are developing understanding. But its steps can be collapsed into one of the other methods E, F, G as students move to larger numbers.

Which written methods meet more criteria of methods that should be emphasized? Method D is the only method that can be undertaken from the left (as well as from the

FIGURE 2: *Multidigit Addition Methods that Decompose into Base Ten Units*

Place value drawing for all methods

$$\begin{array}{r} 456 \\ +\,167 \\ \hline \end{array}$$



General methods for 2- and 3-digit numbers

D.
$$\begin{array}{r} 456 \\ +\,167 \\ \hline 500 \\ 110 \\ 13 \\ \hline 623 \end{array}$$

E.
$$\begin{array}{r} 456 \\ +\,167 \\ \hline {}_{1\,1}\ \ \\ 623 \end{array}$$

F.
$$\begin{array}{r} {}^{5\,6}\ \\ \cancel{4}\cancel{5}6 \\ +\,167 \\ \hline 623 \end{array}$$

G.
$$\begin{array}{r} {}^{1\,1}\ \\ 456 \\ +\,167 \\ \hline 623 \end{array}$$

Methods E, F, and G generalized to 6-digit numbers

E.
$$\begin{array}{r} 456{,}789 \\ +\,167{,}189 \\ \hline {}_{1\,1}\ \ \ {}_{1\,1}\ \ \\ 623{,}978 \end{array}$$

F.
$$\begin{array}{r} {}^{5\,6}\ \ \ \ {}^{8\,9}\ \\ \cancel{4}\cancel{5}6{,}\cancel{7}\cancel{8}9 \\ +\,167{,}189 \\ \hline 623{,}978 \end{array}$$

G.
$$\begin{array}{r} {}^{1\,1}\ \ \ \ {}^{1\,1}\ \\ 456{,}789 \\ +\,167{,}189 \\ \hline 623{,}978 \end{array}$$

Note: Methods E, F and G are all variations in the standard algorithm, but Method E is conceptually clearer and easier.

right) so this is an advantage initially when building understanding. Method D also shows the place values explicitly, so this also makes it clearer that one is adding like multiunits. Method E has several advantages, especially compared to Method G, and supports place value understanding and use by:

- making it easier to see the teen sums for the ones (13 ones) and for the tens (12 tens), rather than separating these teen sums in space as in Method G so that it is difficult to see the 13 or the 12;
- allowing students to write the teen numbers in the usual order as 1 then 3 (or 1 then 2) instead of, as in Method G, writing the 3 and then "carrying" or grouping the 1 above;
- making it easier to see where to write the new 1 ten or 1 hundred in the next left place instead of above the left-most place (a well-documented error that arises more with problems of 3 or more digits and is easier to make when one is separating the teen number as in Method G); and
- making it easier to write the new 1 on the line above exactly the correct (next left) column; when one writes the 1 above the addends in Method G the 1 is spatially separated farther.

It is easier to carry out the single-digit additions with Method E because you just add the two larger numbers you see and then increase that total by 1, which is waiting below. In Method G, students who add the two numbers in the original problem often forget to add the 1 on the top. Many teachers emphasize that they should add the 1 to the top number, remember that number and ignore the number they just used, and add the mental number to the other number they see. This is more difficult than adding the two numbers you see and then adding 1. Method F adds the 1 into the top number instead of writing it above. This makes it easy to add the two numbers that are there, but some students get this method confused with subtraction because you are crossing out a number in the top. In Methods F and G, you change the problem by modifying the top number. With Method E, the two multidigit addends and the sum are all in their own spaces, which is conceptually clearer.

Method E meets more criteria as an emphasized method, so it can be introduced in Grade 1 (if no students develop it) along with Method D, which can move from the left and helps students see the values of the places. Even though Method G has many disadvantages, many parents are familiar with it, so it is useful to discuss and explain it in the classroom and relate it to other methods. Method F makes the addition easy to carry out, but it does change the problem and is easily confused with subtraction, so it might be better to avoid it unless students develop it themselves.

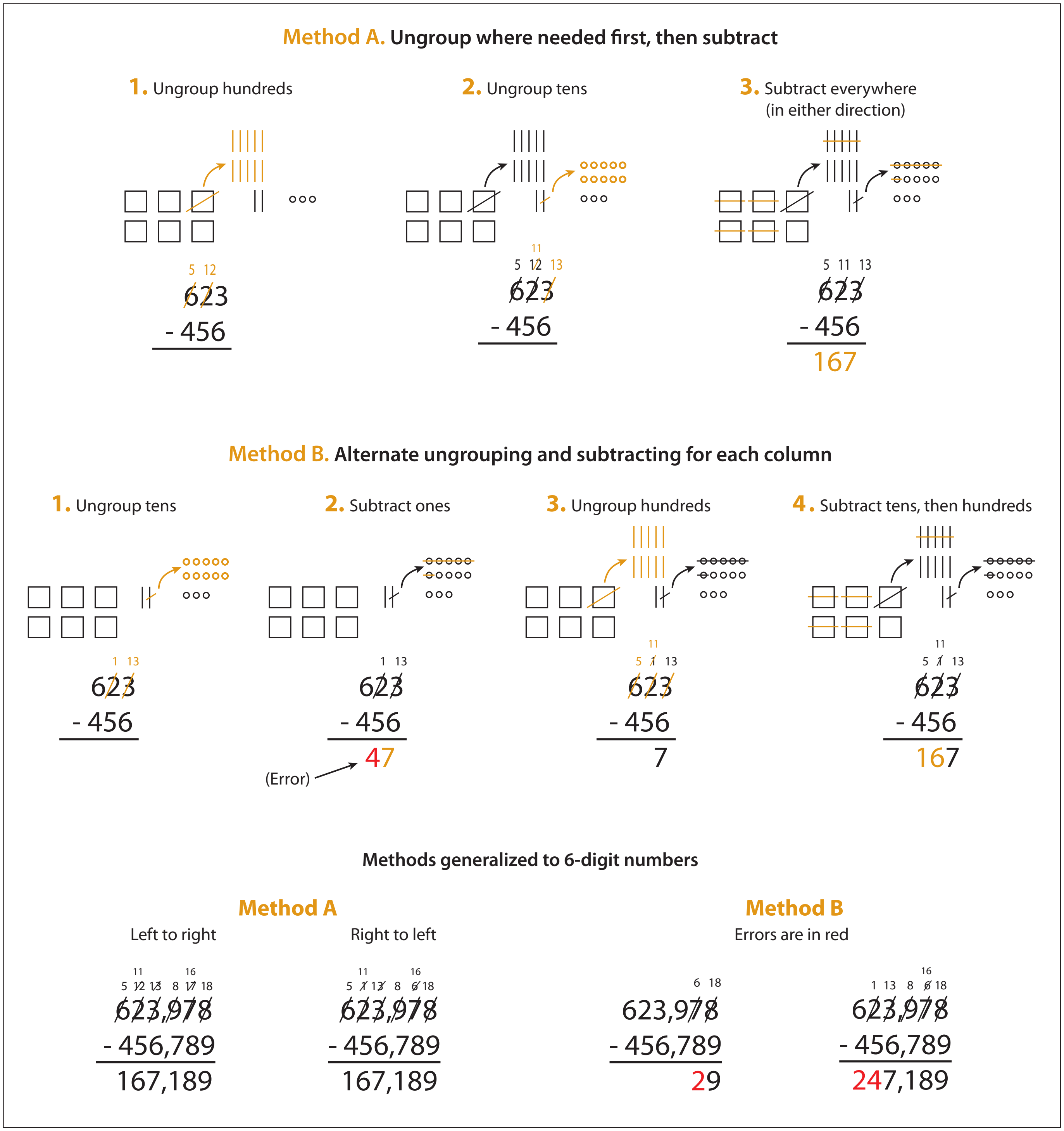
Multidigit Subtraction

Multidigit subtraction approaches that begin with one undecomposed number can involve counting or adding on up to the known total or can involve counting or subtracting down. Written methods for the former approach would look like the methods in Figure 1, but the sum is known and the unknown addend is being found. When finding the unknown addend, one monitors when one has reached the known sum and then finds how many were added/counted on. More variations of written methods that begin with one undecomposed number are shown in Fuson, et al. (1997), in Verschaffel et al. (2007), and in NCTM (2010, 2011). But as with addition, these methods are not practical for larger numbers, and some students find them difficult even for 3-digit numbers.

Written methods for the standard subtraction algorithm use the approach of decomposing into base-ten units and are shown in Figure 3. For subtraction, you need to check the number of units in the top number for a given column to see if there are enough of those units to subtract from (e.g., *Is the top number greater than or equal to the bottom number?*). If not, you need to get more of those units by ungrouping one unit from the left to make ten more of the units in the target column. All of these "checking and ungrouping if needed" steps can be done first, either from the left or from the right. Then all of the subtracting can be completed either from the left or from the right. (These subtractions can actually be completed in any order, but going in one direction systematically creates fewer errors). This taking care of all needed ungrouping first is shown as Method A with math drawings for a 3-digit example and then without drawings for a 6-digit number at the bottom to show how it generalizes. Students can stop making drawings as soon as they understand and can explain the steps.

Ungrouping from the left and from the right are shown for the 6-digit example. The only difference between these is in columns with two new units shown for a column.

FIGURE 3: *Multidigit Subtraction Methods that Decompose into Base Ten Units*



Separating the two major kinds of steps involved in multi-digit subtracting is conceptually clear and makes it easier to understand that you are not changing the total value of the top number when you ungroup. You are just moving units around to different columns. Many students prefer to move from left to right, as they do in reading, and productive mathematical discussions can take place as students explain why they can go in either direction and still get the same answer.

The Method B variation involves following the same steps but alternating between ungrouping and subtracting.

Alternating steps is usually more difficult, and this method sets up the common subtraction error of subtracting the top from bottom number when it is smaller (e.g., for 94 – 36, get 62). Even when you know you should check and ungroup if needed, alternating steps prompts errors. For example, in the 3-digit number in Step 2 you have just subtracted 6 ones from 13 ones to get 7 ones. You look at the next column and see 1 and 5, and 4 pops into your head (if you are only in second grade). You write 4 and move left. In the 6-digit problem, the three errors that can be created by alternating ungrouping and subtracting in Method B are in red. Although this alternating method can be used for numbers of any size, it is not as easy or conceptually clear as Method A. For 2-digit numbers, the alternating Method B and non-alternating Method A are the same because there is no iteration of the steps.

This top-from-bottom subtraction error noted above has been very frequent in the past, partly because of the usual practice of introducing problems with no ungrouping (e.g., 78 – 43) in Grade 1 and only moving to ungrouping problems a year later, in Grade 2, after students had already solidified a subtraction method that seemed to work well. That subtraction method involved looking at a column and subtracting the two numbers you saw there regardless of their relative size. It is our hope that the CCSS-M will result in the elimination of this common textbook practice because no general 2-digit subtraction methods (with or without ungrouping) are included in the Grade 1 standards. Therefore, in grade 2, subtraction with ungrouping can be addressed first so students learn to check from the beginning to see if they need to ungroup each column. This initial understanding of subtracting as possibly needing ungrouping, combined with reduced use of the alternating method, may greatly increase understanding of subtraction methods and contribute to a greater number of correct answers.

There are also strategies that only apply easily to some numbers. For example, for 98 + 47, a student may think, "I give 2 from the 47 to the 98 to make it 100, and then the 47 is 45, so I add 45 to 100 which is 145." This strategy recomposes both numbers to make a particularly easy addition. The subtraction counterpart to this strategy is more difficult for students (and some teachers) because one must know and remember to keep the difference for the original and the new problem the same: 145 – 98 has the same difference as 147 – 100, which is 47. Exploration of such limited strategies can support understanding of relationships between addends and sums in addition and subtraction, but such work cannot replace the extensive time needed to develop understanding of written methods for the standard algorithm and moving them to fluency. Also, it is not the case that learning the standard algorithm prevents students from discussing good strategies for very special cases, such as 398 + 427. Standard algorithms and special strategies are mathematical tools that students can learn to apply strategically. Asking, *Can this method be used for all numbers of a given size or be extended to larger numbers?* is a good mathematical practice in the classroom.

Multidigit Multiplication

The key NBT standards for multidigit multiplication and division and all operations on decimals and the related excerpts from the grade-level introductions to the critical areas from the CCSS-M are in Table 2. Multidigit multiplication does not deal with as many places as multidigit addition and subtraction, where 6 places may be involved as opposed to 5-digit products, and the learning path is shorter. Multiplication moves from the first year (in Grade 4) where the approach of the standard algorithm is developed and explained using visual models (diagrams) to the second year (in Grade 5) where the approach of the standard algorithm continues to be deepened and then is used fluently.

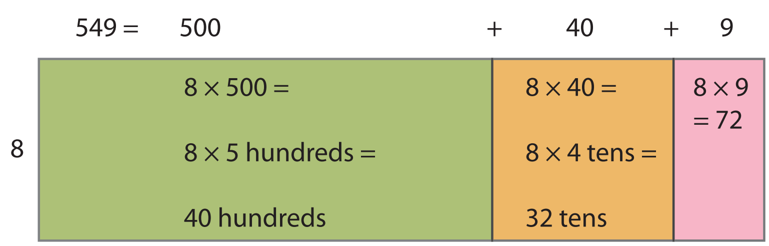
The major issue for multidigit multiplication is what to multiply by what and how the place values of the digits in the factors affect the place values of the partial products. An array or area model can help students understand these issues in terms of how the partial products are recorded. Figure 4 is modified slightly from the NBT Progression document and shows area models, distributive property equations addressing place value, and methods for recording the standard algorithm with 1-digit multipliers. These help students understand how each partial product comes from a multiplication of a kind of unit in one number times a kind of unit in the other number. The Methods A and B can be abbreviated to Method C where the partial products are written within the product space in one row rather than as separate rows that show the place values, but this method is more complex. Methods A and B are conceptually clearer.

Figure 5 shows how the place values in several methods for recording the standard algorithm for 2-digit by 2-digit multiplication relate to the place values in an area or array model and to each other. Methods D, F, and G write all four partial products, while Method E abbreviates the products of a given number into one row as did Method C. However, in Method E, partial products can be seen as diagonals written as for Method E in Figure 2 in addition (e.g., 24 ones is a small 2 in the tens place and a 4 in the ones place and likewise for the 54 tens, 12 tens, 27 hundreds). Writing these below allows students to see these products, and it puts all of the carries (regroupings) in the correct place. In another variation of this abbreviated method, shown in Figure 6 on the left as Method H, the 1 carried above the tens column is from 30 x 4 = 120, so it is actually 1 hundred and not 1 ten. It is confusing to have it in the tens column. Furthermore, having the carries above disconnects them from the rest of their product, so the steps and meanings of the digits can get confused. This method also alternates multiplying and adding, increasing its difficulty even further. This should not be an emphasized method but might be discussed if students bring it into the classroom. Method C also has a variation in which the carries are written above instead of below, which changes the problem, and makes it difficult to see the products because they are separated physically.

Multidigit multiplication can be a challenging visual-spatial task. Some students find it difficult to multiply without an area or array model. They prefer to make a quick area sketch, write the products inside, and then add up the

FIGURE 4: *Written Methods for the Standard Multiplication Algorithm, 1-digit × 3-digit*

Array/area drawing for 8 × 549



$$8 \times 549 = 8 \times (500 + 40 + 9)$$
$$= 8 \times 500 + 8 \times 40 + 8 \times 9$$

Method A:

Left to right showing the partial products

	thinking:
549	
× 8	
4000	8 × 5 hundreds
320	8 × 4 tens
72	8 × 9
4392	

Method B:

Right to left showing the partial products

	thinking:
549	
× 8	
72	8 × 9
320	8 × 4 tens
4000	8 × 5 hundreds
4392	

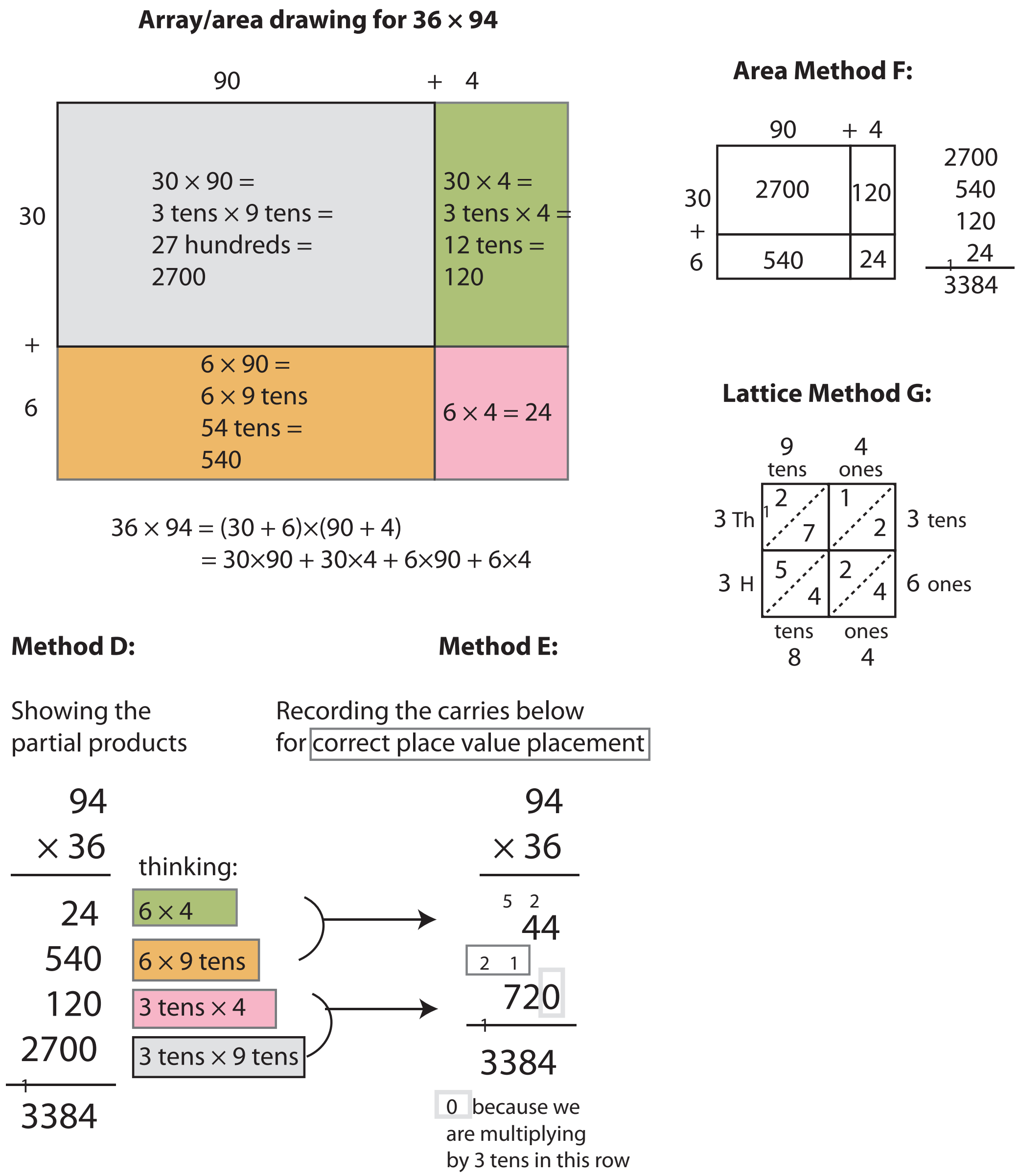
Method C:

Right to left recording the carries below

```
  549
×   8
 437
  022
 4392
```

Method A proceeds from left to right, and the others from right to left. In Method C, the digits representing new units are written below the line rather than above 549, thus keeping the digits of the products close to each other, e.g., the 7 from 8×9=72 is written diagonally to the left of the 2 rather than above the 4 in 549.

FIGURE 5: *Written Methods for the Standard Multiplication Algorithm, 2-digit × 2-digit*



Written Methods D and E are shown from right to left, but could go from left to right. In Method E, digits that represent newly composed tens and hundreds in the partial products are written below the line intead of above 94. This way, the 1 from 30 × 4 = 120 is placed correctly in the hundreds place and the digit 2 from 30 × 90 = 2700 is placed correctly in the thousands place. If these digits had been placed above 94, they would be in incorrect places. Note that the 0 in the ones place of the second line of Method E is there because the whole line of digits is produced by multiplying by 30 (not 3).

products outside as in Area Method F in Figure 5. Such a written method might be a little too long for fluency with the standard algorithm because it involves a drawing, though it shows place values clearly, can generalize, and is efficient. But it also seems better even in Grade 5 to allow some students to use Method F with accuracy than to use the more abstract partial products Method D if students are likely to make errors with this more abstract recording.

FIGURE 6. *Further Written Methods for the Standard Multiplication Algorithm, 2-digit × 2-digit*

Method H:

A misleading abbreviated method

$$\begin{array}{r}
1 \\
2 \\
94 \\
\times\ 36 \\
\hline
564 \\
{}^{1}282 \\
\hline
3384
\end{array}$$

From $30 \times 4 = 120$. The 1 is 1 hundred, not 1 ten.

Helping Steps Method I:

$$\begin{array}{rcl}
94 & = & 90 + 4 \\
\times\ 36 & = & 30 + 6 \\
\hline
30 \times 90 & = & 2700 \\
30 \times 4 & = & 120 \\
6 \times 90 & = & 540 \\
6 \times 4 & = & 24 \\
\hline
 & & {}^{1} \\
 & & 3384
\end{array}$$

The lattice method of multiplication is hundreds of years old and has become popular in some instructional programs. In Figure 5 we can see how this Lattice Method G is related to the area model. The products of each digit in each factor are written within the area model squares. Diagonals are drawn within each such square to locate the tens and ones of each partial product coming from the multiplication of two single digits. These diagonals inside the whole area square have place values that move from ones to thousands from the bottom right to the bottom left and then up to the top left. These place values are derived from the patterns for multiplying place values (e.g., tens times tens is hundreds). We label the place values of the factors and diagonals in the lattice multiplication so that we can see how the place values in each partial product relate and align. If this method is used in the classroom, it is important to emphasize these place values, so that students understand what they are doing, and are not rotely memorizing a procedure. That is not a CCSS-M approach.

Method I in Figure 6 is a "helping step" version of Method D developed by a class of Grade 4 students from a low SES school. These students recognized that many of them were making mistakes using Method D, and they developed this method to help eliminate these mistakes. By writing out the tens and the ones in each factor, they could see the number of zeros, and thus use the patterns involving tens and hundreds more easily. (The partial products Method D and the Area Method F also show the place values in the factors). They wrote the biggest product first so they could correctly align the other products under it, which also has the additional advantage of showing the approximate size of the full product rapidly. They wrote out the factors of each partial product because some students were not systematic in the order in which they multiplied, and by writing the factors for each partial product, they could check on whether all partial products were included. These steps also supported student efforts to explain each step in the method, initially relating it to an area or array model but eventually omitting this model. Recording all of these steps may be too extensive for fluency with the standard algorithm, but students did stop recording particular steps as they no longer needed their support, thus moving toward the greater fluency of Method D.

Method D can be undertaken from left to right, as can Method I, so Method I can collapse to the left to right version of Method D. Area Method F and Method D are the conceptually clearest methods, and Method D is fast enough for fluency.

Multidigit Division

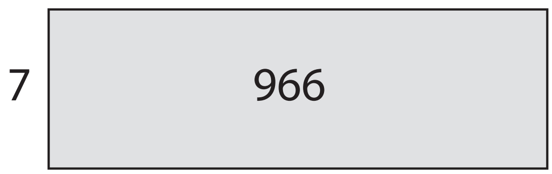
Like multiplication, multidigit division does not deal with as many places as does multidigit addition and subtraction, and again, the learning path is shorter. Developing fluency with multidigit division takes three years because students first develop and explain the approach of the standard algorithm with visual models for dividing by one-digit numbers in Grade 4 and then extend the approach in Grade 5 to dividing by 2-digit numbers where the difficulties of estimating complicate division. In Grade 6 the standard algorithm is used fluently for one- and two-digit divisors.

Array and area models can support understanding of strategies for division. Figure 7 shows Method A including both the recording of the numerical calculations and the area model that corresponds to those calculations as shown in the NBT Progression document. The full multiplier of the divisor at each step in Method A is written above the dividend so that students can see the place value and make a clear connection to the place values in the area

FIGURE 7: *Written methods for the standard division algorithm, 1-digit divisor*

Area/array drawing for 966 ÷ 7

? hundreds + ? tens + ? ones

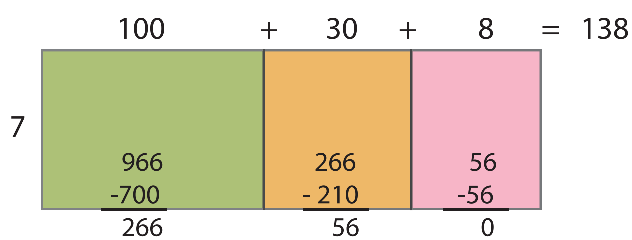


$$7\overline{)966}$$ with ??? above the dividend

Thinking: A rectangle has area 966 and one side of length 7. Find the unknown side length. Find hundreds first, then tens, then ones.

$$\begin{aligned} 966 &= 7\times100 + 7\times30 + 7\times8 \\ &= 7\times(100 + 30 + 8) \\ &= 7\times138 \end{aligned}$$

Method A:



8
30
100
(8 + 30 + 100 = 138)
7)966
- 700
266
- 210
56
- 56
0

Method B:

138
7)966
- 7
26
- 21
56
- 56
0

Conceptual language for this method (all numbers below 966 are in square units):

Find the unknown length of the rectangle;
first find the hundreds, then the tens, then the ones.

The length gets 1 hundred (units); 2 hundreds (square units) remain.

2 hundreds + 6 tens = 26 tens (square units).

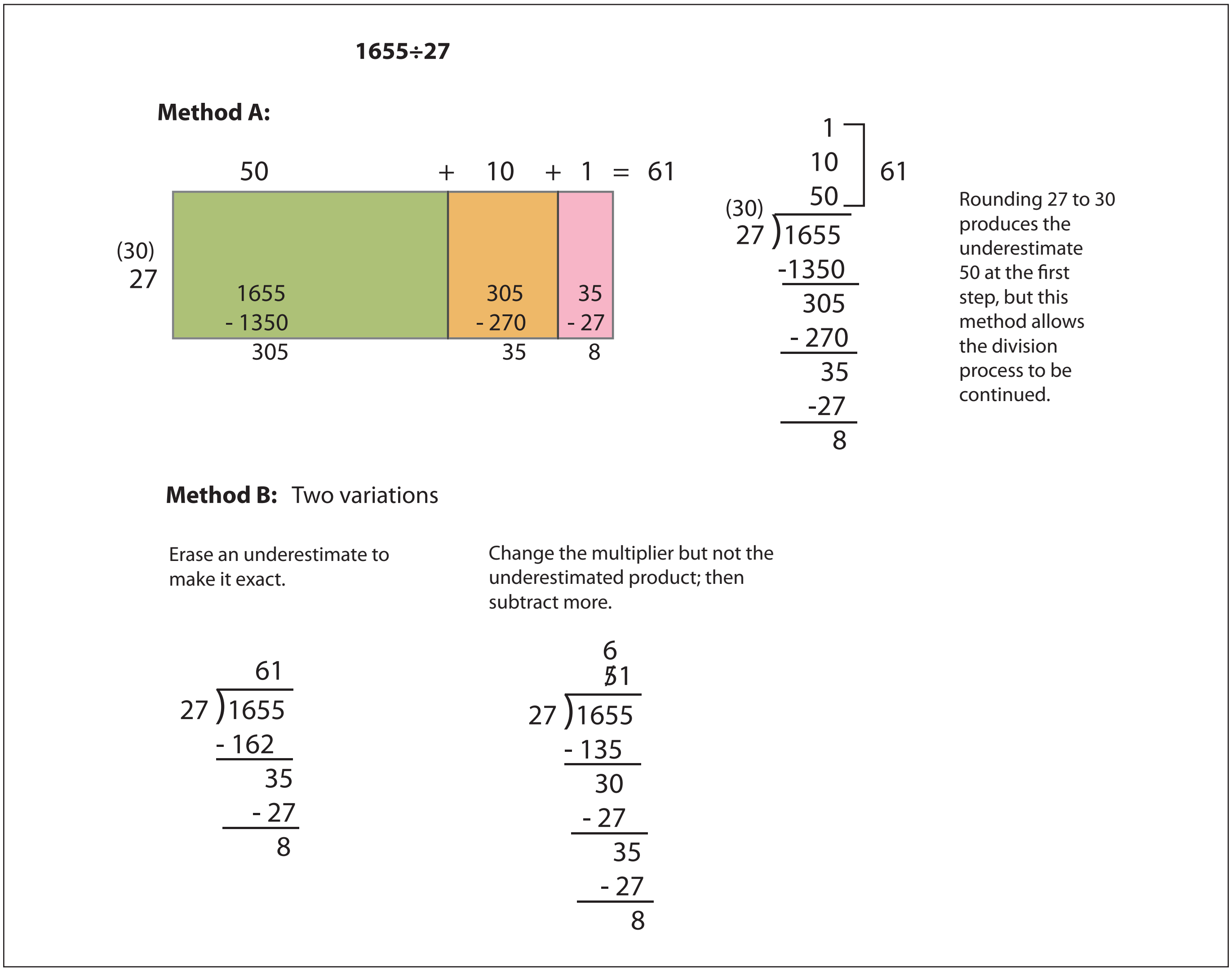
The length gets 3 tens (units); 5 tens (square units) remain.

5 tens + 6 ones = 56 ones (square units).

The length gets 8 ones; 0 remains.

The "bringing down" steps represent unbundling a remaining amount and combining it with the amount at the next lower place.

FIGURE 8: *Written methods for the standard division algorithm, 2-digit divisor*



model. The full product is written at each step, and the amount of the dividend not yet used is also written in full. After experience connecting it to a drawing, Method A can also be undertaken without a drawing as a standard algorithm when calculating quotients that are whole numbers or decimals.

Another written method for this standard algorithm is also shown as Method B in Figure 7. In this method, the zeros are not written in the multipliers or in the partial products within the problem. Digits are brought down within the problem one at a time. Method B makes it more difficult to make the connections to the meaning of the computation, and for that reason, we included conceptual language to communicate the underlying meaning of each step of the calculation. But this method does show clearly the single-digit calculations that are used, and these single-digit calculations are in their place-value locations, as indicated in Method A. Method B becomes important when a quotient has many places, for example, when explaining why decimal expansions of fractions eventually repeat (8.NS.1). Some students may be able to understand and explain Method B right away, and many students can move to it from strategies like Method A, because the move is small. There are also other written methods for the standard algorithm that are variations in between the two shown in Methods A and B. For instance, you could write the zeros on the top but not within the subtractions, or vice versa. Students might develop and use any of these variations.

Method A has a further advantage when dividing by 2-digit numbers because it can involve the use of reasonable estimates. The example shown in the NBT Progression document is depicted in Figure 8 and demonstrates how an estimate of the quotient can be used and then adjusted. Underestimates can be repaired in any place by subtracting another partial product for that place (and adding another rectangle to the area model when first using models). Such a simple repair is an acceptable written method for the standard algorithm. However, some introductions to division allow students to dramatically underestimate the quotient, and as a result, they then have many partial products (e.g., multipliers of 10 + 10 + 10 + 10 + 10) that are used to adjust the estimate. Many extra multipliers and partial products are not consistent with the fluency expectations of the standard algorithm, so students need to be encouraged to be brave and use a multiplier as close as possible to the largest multiplier, for the sake of efficiency. With Method B students cannot continue on from a low estimate. They need to be exact, which often means erasing their underestimate or overestimate. Of course, a student could leave the product and difference they already found, cross out their low multiplier, increase it by 1 or 2, and take away that partial product as another step as in Method A. (See the second variation in Figure 8). Note that overestimates are still best fixed by erasing and trying a lower partial quotient because repairs are difficult to carry out correctly.

Standard Algorithms for Operations on Decimals

The CCSS-M emphasize explaining operations for whole numbers using models that highlight the place value quantities and their roles in the operation. These understandings form the basis for operations with decimals in Grade 5 where students are also expected to explain these operations with models that highlight the place value quantities and their role in these operations. The mathematical approaches of the standard algorithms for whole number addition and subtraction that involve adding and subtracting like place value units, composing or decomposing where needed, also apply to the addition and subtraction of decimals. The lines of reasoning for whole number multiplication and division also extend to decimal multiplication and division. The extensions of these written methods for whole number computation to decimal computation are discussed in the NBT Progression document.

Roles of Leaders

Leaders in mathematics education have vital roles to play with respect to CCSS-M computation. They need to understand deeply and be able to explain the middle ground laid out in the CCSS-M so that they can lead teachers, parents, and administrators out of the sometimes deeply entrenched positions created during the "math war" years. This requires helping everyone understand that standard algorithms are to be understood and explained and related to visual models before there is any focus on fluency. The models help to build understanding for methods that highlight place value understandings and properties of operations. Full fluency is not achieved until subsequent years.

The real problem is with how the standard algorithms have been seen in the past—as fixed written methods learned rotely rather than as a mathematical approach based on a big idea that can be played out using various written methods. Furthermore, the recognition that easier or more meaningful methods can be chosen for emphasis in the classroom has often been lacking.

Leaders can help everyone understand that becoming fluent with the standard algorithm entails using a sensible written method that implements the mathematical approach of the standard algorithm. The mathematical approach of the standard algorithms is distinguished by a big mathematical idea—that multi-digit calculations can be reduced to single-digit calculations while at the same time attending to the placement of these digits by attending to their place values.

Leaders can highlight the power of this big mathematical idea and the importance of allowing written methods that empower students with sensemaking and a deeper understanding of the base-ten system and properties of operations as well as computational skill as they build fluency with standard algorithms. Although these written methods may look different from those that parents or teachers learned when they were in school, they nevertheless implement the deep mathematical ideas that are encapsulated in the standard algorithms.

Leaders need to be able to explain, for each operation, how visual models help explain important aspects of the place value and properties of operations in each of the written steps of the standard algorithm. They need to understand different variations in how to record these steps, as discussed in this paper and in the NBT Progression document. They need to be able to compare and discuss different variations, with attention to possible advantages and disadvantages of each variation at different points in a student's learning. They need to be able to explain methods that may be brought into classrooms by students that may provide yet an additional variation that makes sense mathematically. They should also know ways to connect written methods to more advanced notation that uses place value and properties of operations, such as in this string of equations:

$$\begin{aligned} 8\bullet 549 &= 8\bullet(500 + 40 + 9) \\ &= (8\bullet 500) + (8\bullet 40) + (8\bullet 9) \\ &= (8\bullet 5)\bullet 100 + (8\bullet 4)\bullet 10 + 8\bullet 9 \end{aligned}$$

They should recognize that the expression on the last line encodes the approach of the standard multiplication algorithm, that it could be evaluated from left to right or right to left (after evaluating each term), and that such work is much like multiplication with polynomials, which students will learn in later grades. Such equations do not need to be used by students, but they can be helpful for teachers in seeing the big idea in action, and being able to relate it to mathematics that will come later for their students. Leaders need to use all of these understandings to support teachers, students, parents, and administrators in the development of their own understandings.

Above all, leaders need to help others see this CCSS-M conceptual approach to computation as deeply mathematical and as enabling students to make sense of and use the base ten system and properties of operations powerfully. How the regularity of the mathematical structure in the base ten system can be used for so many different kinds of calculation is an important feature of what we want students to appreciate in the elementary grades. The relationships across operations are also a critically important mathematical idea. The CCSS-M focus on understanding and explaining such calculations, with the support of visual models, enables students to see mathematical structure as accessible, important, interesting, and useful. This is the value of including the meaningful development of standard algorithms in the CCSS-M.

Conclusion

Various written methods that reflect the approach of the standard algorithm for each operation, as well as other general approaches, have been shown and discussed in this article. Some written methods are easier to understand or carry out and therefore should be introduced to students. Other written methods may be introduced by particular programs for various reasons. All variations are interrelated, and it is important for variations in written methods to be explored and discussed by students. Discussing and relating and explaining variations is also an important use of the Standards of Mathematical Practice found in the CCSS-M.

Our examples have attempted to carve out a middle ground for acceptable written methods that reflect the core mathematical approach of, and therefore can be considered to be, standard algorithms. We believe that methods with steps that show the crucial components of the standard algorithms (e.g., showing four partial products for a 2-digit x 2-digit multiplication) are acceptable versions of the standard algorithms but methods with many extra steps that make them less efficient are not acceptable versions.

We also believe that variations with many extra steps can be pedagogically useful as students progress toward the standard algorithms because they help students see and discuss the place value units and the properties of operations, with the extra “helping steps” being dropped by individual students as they progress toward fluency. However, because the CCSS-M emphasize understanding and explanation as the basis for moving to fluency, and some students may be moving more slowly along their learning trajectory than others, some students may continue to use visual models or a more extensive written method for the standard algorithm for some extra period of time.

Finally, we believe it is important for there be time and space in different instructional programs for the development of different written methods that support understanding and the development of fluency with the standard algorithms. These approaches and methods should be related to visual models that reflect the mathematical approach of the standard algorithm, and when written without visual models, can be extended to larger numbers and to decimals in later grades. These approaches and methods should also be able to be written systematically in ways that are clear and not misleading. We also believe that even with the richness of materials in some of our currently existing instructional materials, we may not yet know of some written methods that can also be useful to explore as we address the expectations of the CCSS-M. For this reason, we will have ongoing discussion of advantages and disadvantages of various written methods at the Mathematics Teaching Community, https://mathematicsteachingcommunity.math.uga.edu/ under the standard-algorithms tag.

References

Fuson, K. C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and Instruction*, 7, 343-403.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning.* New York: Macmillan, pp. 243-275.

Fuson, K. C., & Li, Y. (2009). Cross-cultural issues in linguistic, visual-quantitative, and written-numeric supports for mathematical thinking. *ZDM Mathematics Education*, 41, 793–808.

Fuson, K. C., Wearne, D., Hiebert, J., Human, P., Murray, H., Olivier, A., Carpenter, T., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers at work in addition and subtraction. *Journal for Research in Mathematics Education*, 28, 130-162.

National Council of Teachers of Mathematics (NCTM) (2010). *Focus in Grade 1: Teaching with Curriculum Focal Points.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM) (2011). *Focus in Grade 2: Teaching with Curriculum Focal Points.* Reston, VA: NCTM.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* Washington, D.C.: National Academy Press.

National Governors Association Center for Best Practices, Council of Chief State School Officers (2010). *The Common Core State Standards Math.* Washington, D.C.: Author.

Reys and Thomas (2011). Standards for computational fluency: A comparison of state and CCSS-M expectations. *NCSM Journal*, 13 (2), 21 – 32.

The Common Core Writing Team (7 April, 2011). *The NBT Progression for the Common Core State Standards.* commoncoretools.wordpress.com

Verschaffel, L., Greer, B., De Corte, E. (2007). Whole numbers concepts and operations. In F. K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 557-628). Charlotte, NC: Information Age Publishing.