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The Importance of Context in Presenting Fraction Problems to Help Students Formulate Models and Representations as Solution Strategies

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That middle grades and high school students have difficulties solving fraction problems is a common perception of both preservice and inservice middle grades (6-8) and secondary (7-12) mathematics teachers with whom we work. This perception is well founded. Working with fractions, especially multiplication and division within a problem context or in ratio and proportion situations, is difficult for students at many ages. In summarizing research on rational numbers and proportional reasoning, Lamon (2007) articulates that "fractions, ratios, and proportions arguably hold the distinction of being…the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science…" (p. 629).

Understanding fractions, together with solving problems in context (e.g., word problems) and algebraic understanding, is often identified as an area that critically affects student success in mathematics. Wu (2009) argues that, "Because fractions are students' first serious excursions into abstraction, understanding fractions is the most critical step in understanding rational numbers and in preparing for algebra" (p. 8). Confrey and Maloney (2010) further articulate the broad spectrum of these issues as follows:

> There is perhaps no more important conceptual area in mathematics education than rational number reasoning. The basis of the multiplicative concepts field (Vergnaud 1983, 1996), rational number reasoning underpins algebra, higher mathematical reasoning, and the quantitative competence required in science. Failure to develop robust rational number construct reasoning and skills in elementary and middle school continues to plague American students. Rational number reasoning is complex, and master represents cognitive synthesis—understanding, distinguishing among, modeling, and interweaving a remarkable assortment of distinct yet closely related concepts over many years. (p. 968)

These struggles with understanding fractions are in addition to the broader challenges students often face in connecting relationships in word problems and algebraic equations (Keiran, 2007). Recently, similar struggles have been identified among preservice and inservice elementary, middle, and secondary mathematics teachers when asked to model or provide representations-based solutions to fraction word problems, with many only able to provide solutions that are primarily procedural and in symbolic form (e.g., Sjostrom, Olson, and Olson, 2010; Olson and Olson, 2011). These finding raise questions about the extent to which teachers are prepared to address these challenges with their students.

Although mathematical content is important, the context within which the mathematics content is situated is also critically important. In fact, in the Common Core State Standards for Mathematics (CCSS, 2010), the second *Standard of Mathematical Practice* (SMP 2) addresses the importance of students' abilities to contextualize and decontextualize quantitative relationships as follows:

> Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. (CCSS, p. 6).

Such recognition of the importance of contextualizing mathematics is not new, but the way contextualization occurs has historically been the topic of debate. In our discussions, we focus on the point of view offered by Boaler (1993) (drawing on the work of Lave (1988)) who suggested that, "the specific context within which a mathematical task is situated is capable of determining not only general performance but choice of mathematical procedure" (p. 13). It is primarily through this lens that we identify our notion of the importance of context with respect to the problem presented in this article, namely, a problem in which certain mathematical procedures, or in our case, representations and models, arise as primary solution strategies.

For the past several years we have engaged in examining the work of middle grades students as well as preservice elementary, middle, and high school mathematics teachers relative to how they use modeling and representation methods to solve word problems involving fractions (Slovin, Olson, and Zenigami, 2007; Olson, Zenigami, and Slovin, 2008; Olson, Slovin, and Zenigami, 2009; Sjostrom, Olson, and Olson, 2010; Olson and Olson, 2011). Four word problems have been consistently used in these studies, and data have been collected from approximately 30 students in Grade 5, 120 students in each of Grades 6 – 8, 40 preservice elementary teachers, and 40 preservice and inservice teachers of Grades 7 – 12.

In this paper, we analyze the responses of a few selected students, preservice mathematics teachers, and inservice mathematics teachers to one contextualized fraction problem using a case study approach. We analyze how individuals express their understandings using the context of the problem to mathematically model a solution without immediately resorting to decontextualized algorithms and discuss the suggestions we have offered elementary, middle, and secondary mathematics teachers, as well as teacher and school district leaders, for how to encourage students and teachers to recognize and take advantage of opportunities to more robustly develop conceptual and contextual understandings of fraction concepts.

A Contextualized Fraction Problem

One of the four problems consistently used in our research is the following contextualized problem called the Painting Problem: *It takes 3/4 liter of paint to cover 3/5 m^2. How much paint is needed to paint 1 m^2? Explain your reasoning and justify your answer.*

Before reading responses of students and teachers given in the article, consider the following prompts and questions as you contemplate finding a solution to this problem:

1. What does a solution to this problem look like? What would be considered a model for the problem situation? What is an equation for the problem situation? How much paint is needed for a 1 square meter board, given the parameters of the context?

2. What work or explanations would we expect to see if this problem was being posed to 5th grade students as their first introduction to such a problem and they have not yet been exposed to fraction computational procedures? What explanations (including mathematical models or expressions) should be provided to the students who are struggling or do not understand what their procedure-based solution "means" in the context of the problem?

3. Suppose this problem was posed to 8th or 9th grade students (say in Algebra or Pre-Algebra) who have not yet fully developed the expected facility with algebra. What explanations (including mathematical models or expressions) should be provided to enable the students to follow the logic and mathematics of the problem and associated discussions so that students are reasonably comfortable with the reasoning underlying the solution? That is, what model or representation, different from an algebraic solution, would likely add to students' comprehension of the mathematics of the problem (i.e., ratio and proportion)?

These questions are given to suggest that students who either have not yet been taught procedures for multiplying and dividing fractions, have little experience working with ratio, or have had these experiences and persist in a state of procedural confusion can meaningfully engage in and solve such problems based only on the context of the problem. In context, students' explanations should be based on making sense of the context regardless of the method of solution. The teachers' challenge, then, is understanding how to provide the support needed so students at all levels make sense of their work and reasoning.

The Painting Problem was selected for two reasons: 1) An accurate model or representation of the problem situation almost directly provides the solution; and 2) The two fractions in the problem have the same numerators and one-to-one functional reasoning stemming from these common numerators seems natural in the context of the problem (e.g., 3/4 L covers 3/5 m^2, or 3 of one thing is "mapped" to 3 of another thing).

Selected Responses to the Painting Problem

In what follows, work samples from two 5th grade students and one 8th grade student are examined and discussed. These work samples illustrate several of the successful models and representations we have seen used in solution strategies to the Painting Problem in earlier research efforts (Olson, Zenigami, and Slovin, 2008; Olson, Slovin, and Zenigami, 2009; Slovin, Olson, and Zenigami, 2007). Work samples from three preservice secondary mathematics teachers are then shared. These work samples demonstrate the range of teacher strategies also found in earlier research efforts (Sjostrom, Olson, and Olson, 2010; Olson and Olson, 2011). In particular, these teacher samples show a range that extends from being able to show a solution, to displaying beginnings of a solution strategy but not fully following to a conclusion, to employing a solution or solution strategy similar to that of the students but more elaborate.

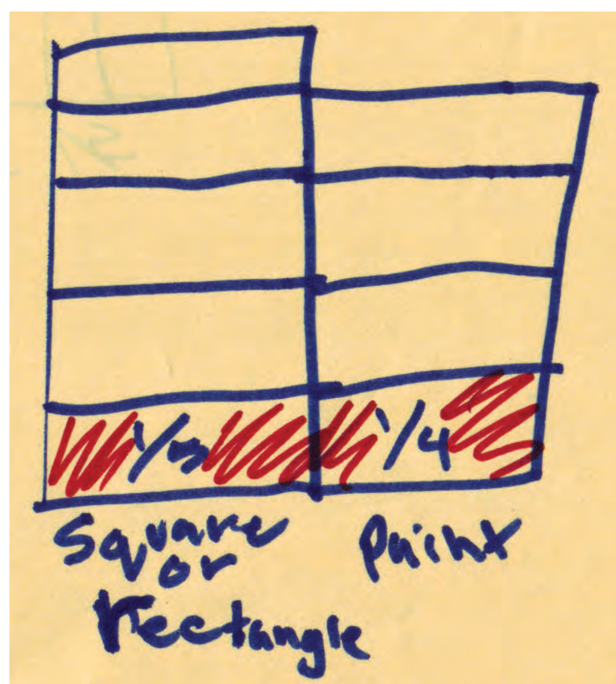
Student Thinking

In Figure 1, Anne[1], a 5th grade student, used area models to represent each quantity (3/4 and 3/5). She drew separate but contiguous area regions to represent each fraction in the problem, and made 1/4 liter and 1/5 square meter the same height, creating unit fraction models derived from the correspondence between 3/4 liter of paint and 3/5 square meter. In her verbal explanation of the model, Anne articulated her use of the one-to-one correspondence between 1/4 liter of paint and 1/5 square meter. Using this correspondence, Anne knew covering 1 square meter (that is, 5/5 square meter) required 5/4 liter of paint.

Anne's original drawing only had the word square listed for the square meter. As she articulated her reasoning, explained her thinking, and justified her conclusion, she felt the need to indicate what she had drawn was a representation of a "square or rectangle." Although she reasoned through the problem using this rectangular representation, she added the words "or rectangle" to the diagram. This reasoning indicated a tension between the object being represented (a *square* meter board) and what was used to represent a square (a rectangular bar).

The impetus for this tension was, again, displayed through her process of justifying her conclusion, and provides evidence of one student's need to verbally align her thinking to her visual model although her visual model is perhaps not a precise representation of the problem situation. These issues of precision in verbal descriptions and visual representations highlight underlying challenges in fostering students thinking with regard to the CCSSM *Standards for*

FIGURE 1: *Anne's Model*



[1] All student names used in this paper are pseudonyms.

Mathematical Practice. In particular, although Anne was not as precise as she could have been with her visual model in representing the problem context, her verbal description eventually did precisely describe her visual model through the process of justifying her answer. Consequently, there are many levels of precision (SMP 6) at play through her process of justifying her conclusion (SMP 3).

Jason, another 5th grade student, used reasoning similar to Anne's but he used different notation (Figure 2). While Jason did not draw a model of the square meter or paint, he verbalized the relationship between 1/4 liter and 1/5 square meter using the correspondence between 3/4 liter and 3/5 square meters, suggesting a mental model of the problem. Jason's thinking led him to the correct solution (5/4 liter) with an appropriate explanation that maintained the one-to-one correspondence until 5/5 square meters (i.e., 1 square meter) was attained. Although Jason's use of the equations 3/4 = 3/5 and 1/4 = 1/5 are not mathematically correct as written, he used these notations as tools to organize his thinking about the one-to-one relationship inherent in the problem. This allowed him to reason through the problem and obtain a correct solution.

Jason's use of imprecise mathematical notation presents another example in which such notation or *symbolic* representation facilitated a student's understanding of the context. This instance once again illustrates the complexity with which teachers will need to approach the implementation of the *Standards for Mathematical Practice*. Precision (SMP 6) is critically important for anyone engaging in mathematical thinking, argumentation, and justification. Additionally, students' emerging visual and symbolic representations must be understood as indicators of their present mental constructs and structures. Consequently, although Jason's symbolic representation proved helpful to him in attaining and justifying a solution, his work also presents an opportunity for his teacher to question Jason to help him reflect on his understanding of the "meaning of the symbols [he chose], including using the equal sign consistently and appropriately" (CCSS, p. 7). That is, Jason's use of the equals sign between 1/4 and 1/5, as well as 3/4 and 3/5 (and 1¼ and 5/5), could potentially be found to be a matter of implementing a "place holder" symbol due to not yet having engaged in discussion and experiences related to ratios and appropriate ratio notations.

Thus, Jason's emerging understandings and mental constructs should not be wholly discounted, nor should the teacher accept at face value his use of the equal sign as implying "equality" simply because it facilitated the accurate answer. Rather, Jason's work must become an opportunity for discussion of the mathematical content, as well as the mathematical practices and representations used to arrive at an answer.

At the time that Anne and Jason were solving this contextualized fraction problem, they had not yet received formal instruction related to division of fractions, ratios, or ratio notation. Yet these students (and others) were able to develop or visualize a model or representation that, in essence, helped them attain a correct solution. That is, creating the model or representation was, itself, a highly

FIGURE 2: *Jason's Model*

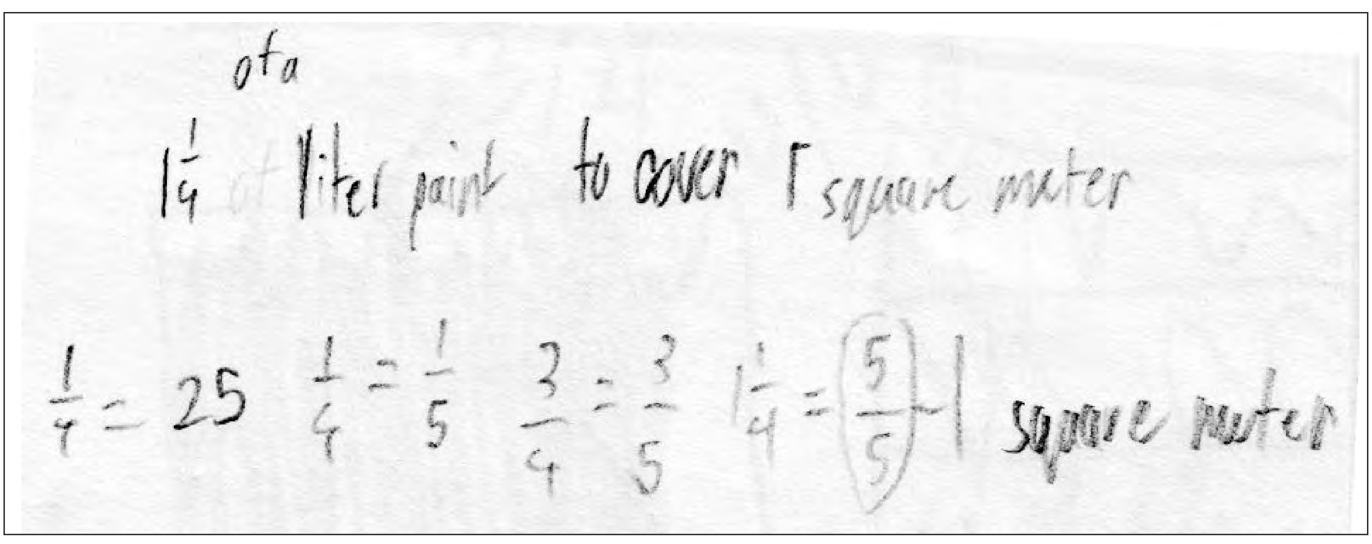
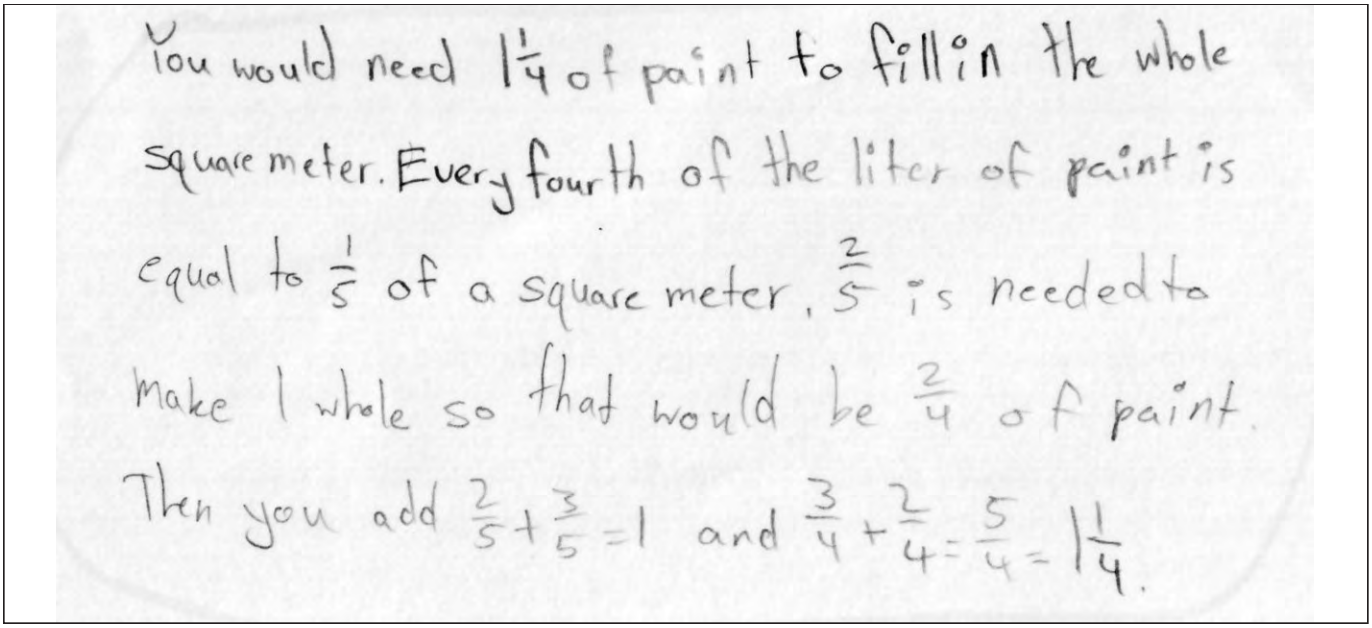


FIGURE 3: *Joseph's Work*



effective solution strategy. In Figure 3 the work of a 6th grade student, Joseph, shows that although he is better able to articulate his thinking, the underlying ideas based on the context of the problem situation are similar to that displayed by the 5th grade students Anne and Jason.

These three students provided thoughtful arguments and justifications (SMP 3), looked for and made use of the structure in the problem context (SMP 7), but displayed varying degrees of precision (SMP 6) in their representations and arguments, if one solely examines their written artifacts.

This kind of problem is not a simple exercise for many middle school students despite having received formal instruction addressing ratio and proportion as well as operations with fractions. In fact, more than half of middle school students asked to solve this problem were not able to do so successfully and very few of these students even attempted a visual model or representation in their effort to find a solution (Olson, Slovin, and Zenigami, 2009). These types of problems have also been found to present teachers with difficulties when asked to provide a visual or other descriptive representation that incorporates minimal symbolic mathematics.

Teacher Thinking

Figures 4, 5, and 6 show the responses of three preservice teachers. In Figure 4, the preservice teacher presents a representation of the fractions in the problem but does not successfully use the model to obtain a solution. This preservice teacher correctly identifies that 2/5 of the square meter is yet to be covered, and attempts to use various equations, but to no avail. In Figure 5, the preservice teacher appears to "know" that more than one liter of paint is needed, and identifies a question that would help solve the problem: "You need what fraction of 3/4 liter is needed to complete this painting?" However, this preservice teacher does not use the representation to reason that 2/3 of the 3/4 liter is needed to complete the square meter. How this reasoning would be useful can be seen in the preservice teachers' use of the model where the second "3/4 liter" is used. The second 3/4 liter covers pieces 4, 5, and 6 of the square meter; however, only pieces 4 and 5 need to be covered (i.e., 2/3 of the 3/4 liter). There is appropriate thinking displayed in the work shown in Figure 5, but the preservice teacher was not able to use the model to find the solution.

In Figure 6, the preservice teacher makes use of the representations to solve and explain the solution to the problem. This preservice teacher does not rely on a "procedure" involving symbolic or algebraic manipulations, but rather, provides reasoning in the context of the relationship between 1/4 liter and 1/5 square meter. This preservice teacher showed an understanding of the problem and used a successful strategy to explain relationships rather than attempting a solution via a rule, such as 3/4:3/5 as 1:x. In

FIGURE 4: *Work of a preservice teacher who is not able to solve the problem*

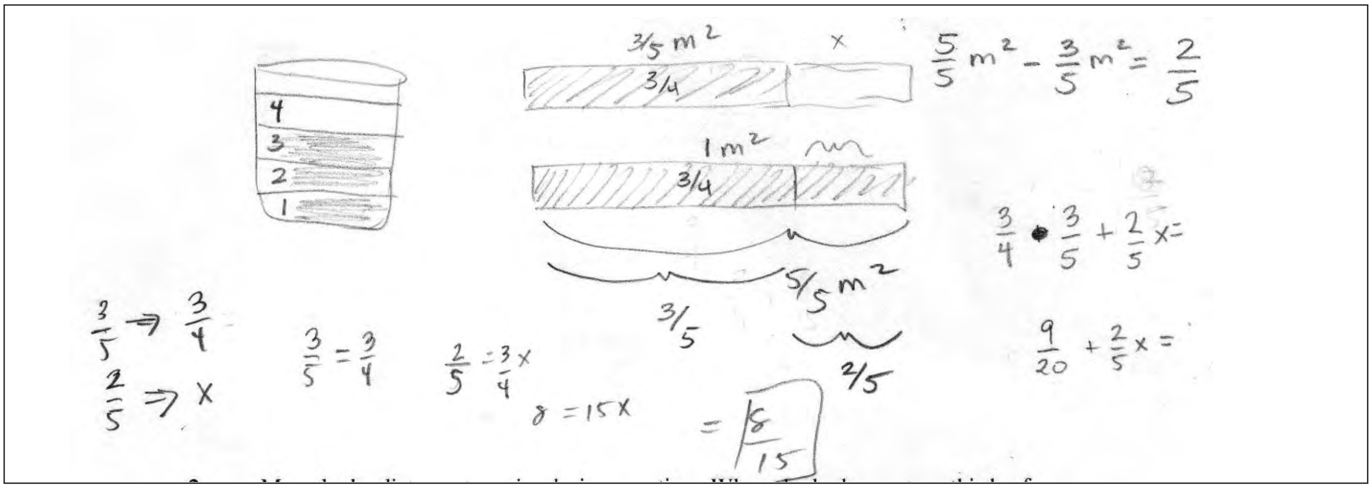


FIGURE 5: *Work of a preservice teacher who employs a model but cannot see how to use it finish solving the problem*

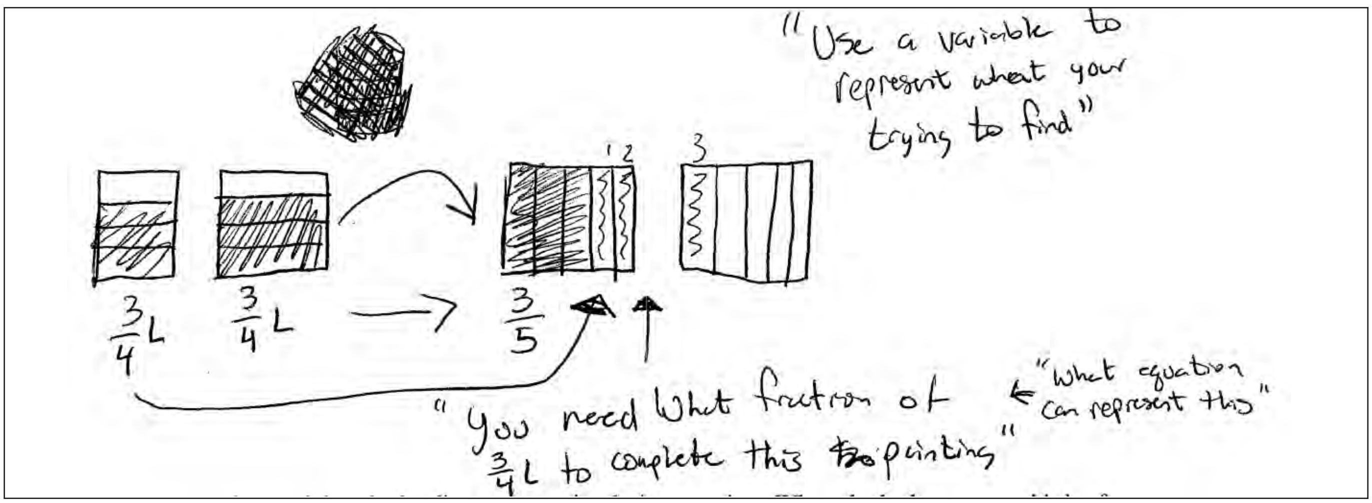
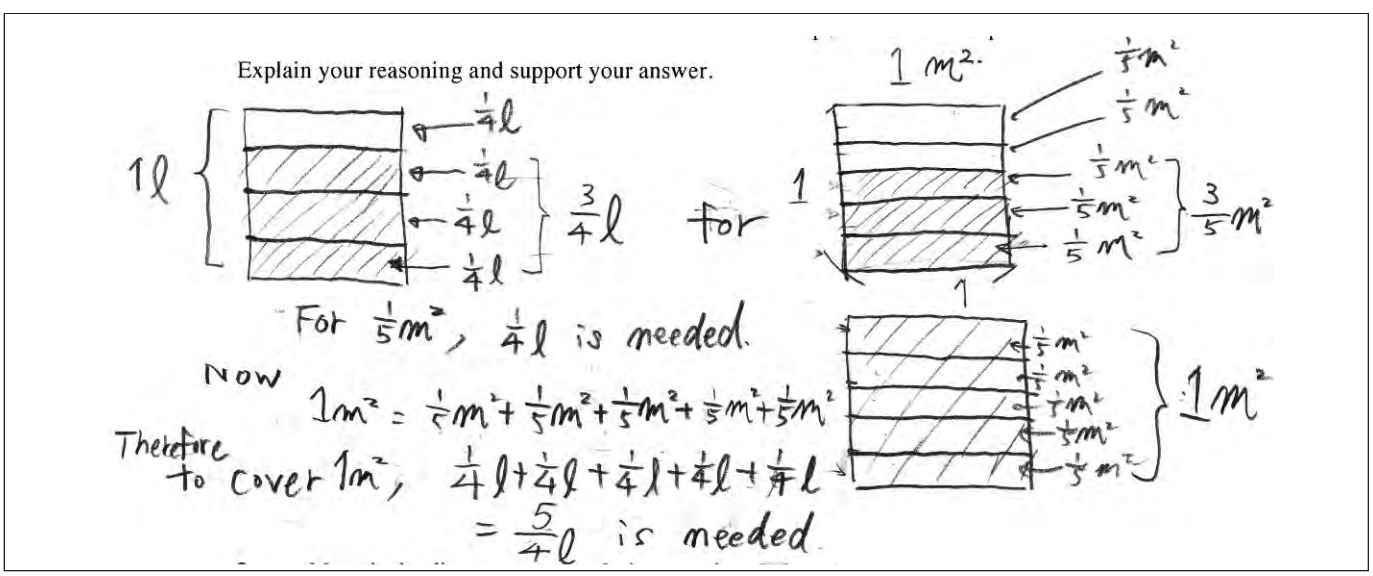


FIGURE 6: *Work of a preservice teacher who uses a model to solve and explain a correct solution*



essence, this teacher's reasoning is similar to that used in the 5th grade students' explanations.

These discussions of teachers' understandings and abilities to represent mathematical contexts are consistent with those of prior research. Such research has shown that teachers appear to not have coherent ideas on how to start thinking about the problem; are able to provide initial thoughts on a solution strategy and use a model up to a point, but do not go further; or are able to provide solution strategies similar to those of the middle school students highlighted in the earlier discussion of student thinking (Sjostrom, Olson, and Olson, 2010; Olson and Olson, 2011).

Commonalities in Student and Teacher Thinking

In the prior research involving responses of approximately 30 students in Grade 5, 120 students in each of Grades 6 – 8, 40 preservice elementary teachers, and 40 preservice and inservice teachers of Grades 7 – 12, when a student or teacher used a visual model or representation to correctly solve the Painting Problem, the model or representation was similar to the examples provided. What was it about the problem or its context that led to the use of a model or representation? Those using a model or representation used a unit rate approach, but the presence of a common numerator may have been instrumental in making that choice, perhaps making visible a one-to-one correspondence between the 1/4 liter of paint and 1/5 square meter. It is our view that the students' use of one-to-one correspondence in the examples shared was fundamentally different from the usual unit rate approach. Importantly, students in Grade 5 and Grade 6 (pre-CCSSM) likely have not encountered ratios in any structural or mathematical sense. Rather, these students (and students like them) move to a "unit numerator" based on the one-to-one correspondence inherent in the problem context – that, if 3 will cover 3, then 1 will cover 1. Thus, the numerators are unitized to make more explicit the one-to-one correspondence, and not as a matter of procedurally unitizing a ratio.

Unfortunately, preservice and inservice teachers often relegate the importance of solution strategies that utilize modeling and representations to the realm of "lesser mathematics." However, the importance of these approaches is well articulated by Wu (2011) in his discussion of a model used to solve a problem involving fractions as follows:

> We see plainly that there is no need to use multiplication of fractions for the solution, and moreover, no need to memorize any solution template. The present method of solution makes the reasoning very clear" (pp. 36-37).

Furthermore, the solutions provided by Anne, Jason, and Joseph exemplify Lamon's (2001) view that, "current instruction in fractions grossly underestimates what children can do without help." (p. 153).

This is not to say that using a model or representation always leads to a correct solution. We saw that some students and preservice teachers were able to create an appropriate beginning model or representation but were unable to finish the problem. A common error involved using common denominators to solve the problem, frequently adding 15/20 (3/4 liter of paint) with 8/20 (the 2/5 square meter left to be painted) to achieve an answer of 23/20. Such solutions suggest a rush to the use of rules and procedures rather than thoughtful use of the context of the problem to find a solution that makes sense.

Conclusion

Campbell, Rowan, and Suarez (1998) argue because algorithms are important, teachers should know and be able to use various strategies for finding a solution, and assist students in making sense of processes and procedures to determine if their work is reasonable. In other words, it is critically important for teachers to "sense-make an algorithm" in various contexts. Through the process of sense making and conceptually understanding algorithms, we argue that teachers' are able to mathematically understand and engage their students' misconceptions. We are not arguing against the importance for teachers and students to be able to symbolically and procedurally arrive at a solution to a word problem involving fractions. However, we suggest that without displaying the ability to understand and use the context of a problem to arrive at a solution through modeling, foundational and conceptual mathematical knowledge is likely not well developed.

It is important to understand how the use of modeling and representations in certain contexts allows for the appropriate conceptual development of key algorithms.

For example, when should the algorithm "flip (invert) and multiply" for division of fractions emerge as contextually making mathematical sense? In Grade 6 standards (i.e., 6.NS.1) when fraction division occurs in a "story context" or by way of "visual fraction models?" Perhaps. Importantly in this CCSSM standard, the context, associated representations, and justification for general (algorithmic) relationships between division and multiplication are all essential components to mathematical sense making.

Additionally, providing a story (or visual) context for which unit rates are computed with respect to division of complex fractions, but only as a solution strategy to this particular contextualized fraction division problem, is arguably a mathematically appropriate context through which teachers can extend students conceptions and misconceptions regarding algorithmic procedures. In such a context, the denominator is inverted and multiplied by the numerator to find a new rate (numerator) per unit (denominator). The CCSSM identifies such a context in Grade 7 (7.RP.1).

Thames and Ball (2010) indicate that, "No one would argue with the claim that teaching mathematics requires mathematics knowledge.…by better understanding the mathematical questions and situations with which teachers must deal, we would gain a better understanding of the mathematics it takes to teach" (p. 221). Furthermore, Keeley and Rose (2006) note that, "Teachers may not be aware of the misconceptions and alternative ideas their students hold, and sometimes, they harbor those very same misconceptions" (p. 6).

There is room for growth in teachers' understandings of the use of context, models, and representations in exploring and solving fraction word problems, particularly with problems that are able to be solved using non-procedural models and representations. Stylianou (2010) indicates that teachers conceptions of representation as a process and practice need further development to include representations more successfully in instruction, especially for non high-performing students. Although there is not uniform agreement on the nature of representations, Stylianou reasons that, "symbolic expressions, drawings, written words, graphical displays, numerals, and diagrams are all representations of mathematical concepts" (p. 326). In this paper, illustrations were provided of how students and teachers both use models or representations effectively as solution strategies.

What is it that we as teachers, teacher leaders, and school leaders can do to help students and teachers reason through problems such as the Painting Problem and use representations or models to assist their thinking? To help students and teachers develop better problem solving abilities related to fractions, we suggest the following.

First, recognize that the development of fraction understanding is a challenge and the way we structure the introduction to the use of fractions to students is very important. Wilson, Edgington, Nguyen, Pescosolido, and Confrey (2011) give an indication of a learning trajectory related to fractions, and indicate that children's early experiences must provide a solid basis for future applications.

Second, recognize the importance of problems in context. As shown in the representations and verbal explanations of Anne and Jason, the words used to situate a problem, if modeled well, provide direction enough so that students can successfully solve the problem. As teachers, the responsibility of including such problems in students' mathematical experiences lies with us. As noted by Sullivan, Zevenbergen, and Mousley (2003) in their discussion of the importance of the context of mathematics tasks, a primary issue in teaching mathematics is that, "teachers need to be fully aware of the purpose and implications of using a particular context at a given time" (p. 111).

Third, reconsider the usefulness of representation and modeling as viable solution strategies. It is important to recognize that students need as much practice in these modeling and representation strategies as they need practice with procedural and algorithmic strategies. The ability to model problem situations and arrive at solutions through the use of those models is not simple or easy to master. With each model or representation used by a student, a teacher needs to practice asking, "How does your model or representation demonstrate what the problem is saying, and how will you use that to help you understand or solve the problem?" Abrahamson (2006) notes the following:

> …one can use these representations without appreciating which ideas they enfold and how these ideas are coordinated. Consequently, learners who, at best, develop procedural fluency with these representations, may not experience a sense of understanding, because they lack opportunities to bridge the embedded ideas, even if these embedded ideas are each familiar and robust. (p. 464)

For this reason, talking about the meaning of the model and represent used in a solution strategy is as important as using it to solve the problem at hand.

In this paper, we shared solutions to a fraction problem and discussed how the context of the problem was useful in finding a solution. We examined the commonalities of the solutions provided by elementary and middle school students and preservice secondary teachers. We discussed how considering the context of the mathematics within problems can be useful for teachers and teacher leaders for helping build situations in which students are asked to model or provide representations for their work. We hope our suggestions are of use to many of you in your mathematics leadership roles.

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