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How Understanding Mathematical Discourses Shapes Principal Noticing

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Abstract

Principal leadership is a key factor in student achievement, but we are not yet sure how knowledge of content influences leadership. Teacher evaluation systems assume principals understand the pedagogical content knowledge (PCK) for multiple disciplines, a particularly challenging expectation for secondary leaders. This study presents a noticing framework of PCK for leadership that describes a progression through four levels, from content-neutral pedagogy to an interconnection between pedagogy and mathematical discourses. Using the framework, the study provides evidence that principals can learn to notice significant mathematical events but may struggle to respond to teachers about those events. The framework can serve as a tool for leaders to learn to notice the role of mathematics in classrooms during their work with teachers.

Keywords: pedagogical content knowledge for leadership, noticing, mathematical discourses, standards for mathematical practice.

Principal leadership is the second-most influential school-related factor impacting student achievement behind teaching (Branch et al., 2013; Leithwood et al., 2004). Although a principal must have a breadth of knowledge about many influences on student achievement, such as hiring quality teachers, aligning curriculum and assessment, and fostering partnerships with parents (Murphy, 2017), this paper examines the knowledge of mathematical content that principals use when they supervise and evaluate classroom teachers.

An important role for school administrators is evaluating teacher effectiveness related to student learning. Teacher evaluation protocols are now available to help principals develop knowledge about what to observe and how to provide feedback for improving instruction. Domain 1 in the Teacher Observation Protocol (Marzano, 2017), Domain 1a in The Framework for Teaching: Evaluation Instrument (Danielson Group, 2022), and four dimensions of The 5D+ Rubric for Instructional Growth and Teacher Evaluation (Center for Educational Leadership, 2016) all describe the importance of specialized content knowledge for teaching. These evaluation protocols also have rubric elements related to ways in which teachers consider disciplinary content in their planning and instructional decisions. These rubrics assume that observers have enough knowledge of the discipline being observed to notice the pedagogical content knowledge (PCK) that teachers draw upon. Middle school and high school principals face particular challenges as the disciplinary content knowledge increases at higher grades. Expecting secondary school administrators to have a sufficient depth of content knowledge to notice PCK and provide related feedback in every discipline they supervise is unrealistic.

The literature does not yet articulate how content knowledge directly influences instructional leadership (Lochmiller et al., 2012), or what principals may need to know about content to support teacher learning and change (Larbi-Cherif, 2016; Lochmiller & Acker-Hocevar, 2016; Steele et al., 2015). Thus, articulating for middle and high school instructional leaders some aspects of the discipline that are essential for students to learn mathematics and helping supervisors notice when those aspects are being enacted may position them to make important leadership decisions that can improve student learning of mathematics.

Purpose

This paper presents a Pedagogical Content Knowledge for Leadership (PCKL) framework that describes a progression from general to content-specific noticing. Using this framework, we describe what nine middle school principals or associate principals attended to during videos of mathematics lessons, and how they said they would respond to the teachers. In our analysis, we demonstrate how the PCKL framework can support principals who are learning to observe mathematical events in classrooms. We conclude with ways that instructional leaders and professional development providers can use the PCKL framework to develop a sharper vision of productive mathematics classrooms and learn to provide feedback targeting student mathematical engagement.

The PCKL framework describes how a leader uses content in a mathematics classroom observation along a continuum: 1) how a classroom event can be observed without considering the content, 2) how the content can be observed within a classroom event but not viewed as important to the event, 3) how instructional decisions can be observed as intersecting with mathematical content, and 4) how mathematical Discourses (Gee, 2011) can be observed as key to the classroom event. Using the PCKL framework, the authors coded the levels of noticing that principals with varying leadership and professional development experience demonstrated when observing math lessons. Our analysis indicates that principals can learn to notice important mathematical events during lessons, but even when they do, they may struggle to provide related feedback to teachers.

Theoretical Frameworks

Discourses and the Standards for Mathematical Practice

Mathematical Discourses (Gee, 2015a) describe the spoken, written, and visual forms of communication that students use as they develop an understanding of mathematics, a sort of disciplinary literacy for mathematics communities (Croce & McCormick, 2020). Big 'D' Discourse captures socially recognizable ways of 'being' within a group, the inextricable ways that members talk and interact, the objects or tools they use, and their values and beliefs. We can quickly discern tourists not only through their language and their cameras, but also through what they wear, how quickly they can pull out the necessary currency, how loudly they speak, or whether they make a faux pas over dinner because tourists have not learned the Discourses that shape the cultural identity of members of the host location. Similarly, we can discern if students have developed a mathematical identity by observing their interactions and behaviors in their classrooms. We act out socially recognizable identities when we use big 'D' Discourses.

The Standards for Mathematical Practice (SMPs) are the first standards listed in the Common Core State Standards for Mathematics (CCSS-M) (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), defining ways in which students act out the socially recognizable identity of an emerging mathematician. By beginning each Standard for Mathematical Practice (SMP) with, "Mathematically proficient students …," the authors of the CCSS-M make clear that how student mathematicians engage with the discipline, the tools they use, the ways they interact, and the language they use matter. The SMPs describe mathematically essential behaviors such as making sense of problems, constructing arguments to justify conjectures, using mathematical tools appropriately, and looking for regularity in repeated reasoning. Whereas the Discourses encompass the plethora of ways that students enact a mathematical identity, the SMPs are descriptors of eight essential Discourses that teachers and leaders can observe. Schools that strive to provide a robust mathematics experience can use the SMPs as a guide for developing a culture of learning in their classrooms, a culture alive with mathematical Discourses.

Noticing

Although researchers have a variety of perspectives about teacher noticing (König et al., 2022), noticing has become an important tool in how researchers understand the work of mathematics teachers. Goodwin (1994), taking a sociocultural perspective, suggested that professional communities can negotiate a shared vision of a knowledge base that is of interest to members of the profession. This perspective of professional vision suggests that noticing is a sociocultural phenomenon, and that one's profession shapes the events attended to and the interpretation of those events.

Teacher noticing is broadly accepted as consisting of three interrelated components: attending to a salient incident, making sense of the incident, and identifying what is important and deciding how to respond (Jacobs et al., 2010; Kaiser et al., 2015; Sherin et al., 2011; van Es & Sherin, 2002). van Es (2011) drew on this understanding of noticing to propose a framework for learning to notice student thinking. The framework shows a trajectory of four levels of noticing student mathematical thinking from a baseline where teachers, "Attend to the whole class environment, behavior, and learning, and to teacher pedagogy," to an extended level where teachers, "Attend to the relationship between particular students' mathematical thinking and between teaching strategies and mathematical thinking." Teachers at the extended level are able to draw connections between instructional decisions and student learning. Moving along the trajectory affords opportunities to learn about the relationship between teacher practice and student understanding.

Researchers have examined how experts notice similarly or differently from novices (Bastian et al., 2022; Huang & Li, 2012; Scholten & Sprenger, 2020). Expert-novice studies found that, through intentional interventions, prospective teachers can learn to attend to salient events and interpret them like more experienced teachers (Jacobs et al., 2022; Miller, 2011; Roth McDuffie et al., 2013). However, teachers who notice student mathematical reasoning at a high level may need additional support in learning how to respond (Jacobs et al., 2010). Targeted professional learning experiences can help both novice and veteran teachers prepare effective responses to the events they deem salient (Jacobs et al., 2022; Jilk, 2016; Sherin & van Es, 2009).

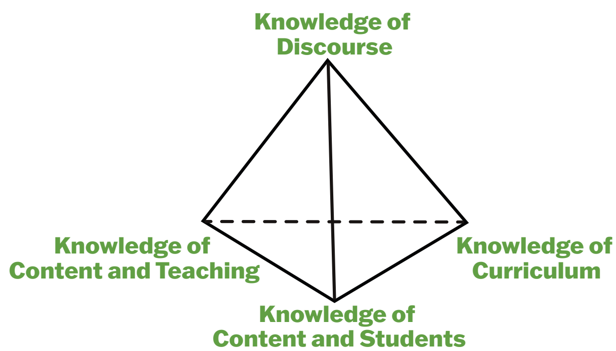
Pedagogical Content Knowledge

In an attempt at articulating standards for professional teachers, Shulman (1987) included PCK - "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 8) - as important among the different bases of knowledge that teaching requires. He continued, "It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction." Hill et al. (2008) articulated PCK as comprised of three parts – knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum.

Knowledge of content and students refers to a teacher's understanding of how students learn mathematics, such as the common mistakes students will make, how they will respond to confusion, or how to draw out student reasoning. Knowledge of content and teaching refers to the teacher's relationship with mathematics, such as how teachers explain math concepts, how they use technology to support student learning, which ideas they deem worthy of further classroom consideration, or which mathematical representations they choose to illuminate the concepts that students are considering. Knowledge of curriculum refers to the teacher's understanding of available materials and when different choices are most appropriate, how to select or modify tasks based on student needs, or whether the trajectory of mathematics through the year and across grades is coherent.

While working with the ideas of PCK, Hauk et al. (2014) described an interplay among the original three-part PCK framework with mathematical Discourses, "the ways of combining and integrating language, actions, interactions, ways of thinking, believing, valuing, and using various symbols, tools and objects to enact a particular sort of socially recognizable identity" (Gee, 2011, p. 29). As shown in Figure 1, Hauk et al. (2014) visualize PCK as extended into a tetrahedron connecting knowledge of Discourse to each of its three components. They defined knowledge of Discourse as "knowledge about the culturally embedded nature of (big D) discourse, including inquiry and forms of communication in mathematics both in and out of educational settings" (p. 171).

FIGURE 1.
Extended Model of PCK



Note: Tetrahedron is used to visualize the relationship between knowledge of Discourse and the three components from the PCK side of the MKT framework. Adapted from "Developing a Model of Pedagogical Content Knowledge for Secondary and Post-Secondary Mathematics Instruction" by S. Hauk, A. Toney, B. Jackson, R. Nair, and J. J. Tsay(2014), *Dialogic Pedagogy, 2(28).*

Leadership Content Knowledge

While the idea of PCK has been understood for the work of teachers, the role of content in the work of school leaders is less clear. Just as teaching and content are inextricably intertwined, Stein and Nelson (2003) showed evidence that there is also a specialized content knowledge for educational leaders that they termed leadership content knowledge (LCK). Although no researcher claims that a principal needs to understand content in the same way that teachers do, some do suggest that the roles of supervision, evaluation, professional development, and resource allocation all intersect with disciplinary content. The disciplinary content knowledge a school leader holds may influence how they enact those roles (Overholt & Szabocsik, 2013; Printy et al., 2008; Stein & Spillane, 2005). Exactly how content influences instructional leadership remains less clear (Lochmiller et al., 2012) as does what they need to know about the disciplines they supervise (Larbi-Cherif, 2016; Lochmiller & Acker-Hocevar, 2016; Steele et al., 2015).

Studies have demonstrated that when administrators learned about characteristics of rich mathematics tasks, high-quality, inquiry-based instruction, or different mathematical representations, they were able to notice student thinking and the link to teacher practices (Boston et al., 2016; Steele et al., 2015). In one study wherein principals were shown the same classroom video in October and again in June, with supported professional development in between, their attention moved from such issues as classroom management, wait time, and which students were called on towards mathematical discourse and student thinking (Nelson & Sassi, 2000). The authors explain, "they need to understand that students' subject-matter thinking is central and that the administrator's 'eye' for classrooms needs to be tuned to following the student thinking in class rather than the teacher's behavior alone" (p. 576). As principals' understandings of what matters in the classroom expanded, what they attended to changed.

Methods

This qualitative study used structured interviews to investigate the professional noticing of mathematics leaders. The participants, methods of gathering data, and manner of coding of interviews are described below.

Participants

The nine participants in this study were chosen based on their role as a middle school principal or assistant principal with at least four years of experience as an administrator. We recognize that the roles of principals and assistant principals have significant differences, but as this study focuses only on the similar work of classroom observation, we refer to all participants as principals. Our participants' experience fell into three categories: (1) three had participated in at least one year of professional development (PD) about research-based instructional practices in mathematics designed for teachers plus a separate component for school leaders, (2) three had taught mathematics at the secondary level but had not had intentional PD about mathematics leadership, and (3) three had neither taught mathematics nor had PD for leading mathematics. One of the principals in the final group had participated in PD designed for mathematics teachers without the additional work on mathematics leadership. Their pseudonyms are listed in Table 1.

Blake, Bradley, and Henry worked in the same school district at different schools, and the other six participants worked for four other school districts in the same region. The school district that offered mathematics leadership PD, located in the Pacific Northwest, has less than 15% of children living below the poverty level and a slightly higher per capita income than the state. More than 80% of children live in English-only homes and 6% live in Spanish-speaking

Table 1: Pseudonyms of Participants by Training Experience.

PD and teaching experience	Pseudonym
PD designed for leaders	Blake Bradley Henry
Former mathematics teachers	Matthew Peter Stuart
PD designed for teachers	Sam
Neither former mathematics teachers nor related PD	Lindsey Warren

homes. Each of the three leaders had participated with their teachers in a three-day workshop about research-based mathematics teaching practices at the beginning of the school year and in five math Studios throughout the year. Studios entailed collaborative planning of a lesson, observing student actions and instructional decisions as the lesson was enacted in one of the teacher's classrooms, and debriefing the lesson (Lesseig, 2016a).

In addition to Studio, principals performed learning walks in each other's buildings. Learning walks involved a group of principals and instructional coaches walking through 3-5 middle school mathematics classrooms for 10-15 minutes each with a brief huddle between classes to discuss what they saw or heard. After the walk, the team of observers categorized any instances of conjecturing, generalizing, or justifying (Lesseig, 2016b) they noticed from students. These three reasoning processes are essential to mathematical sense-making, proof, and problem solving. Distinguishing between when students tentatively believe an idea might be true (making conjectures), see commonalities across cases (generalizing), or build arguments to demonstrate the truth of a statement (justifying) requires close attention to student mathematical reasoning.

The professional development experience used several other frameworks to focus principal attention on student thinking and behaviors. One framework was Weaver's five levels of student discourse (2007) that require increasing levels of cognitive demand: (1) answering, stating, or sharing; (2) explaining; (3) questioning or challenging; (4) relating, conjecturing, or predicting; and (5) justifying or generalizing. Another was student mathematical habits of mind, that in addition to reasoning practices (i.e., conjecturing, generalizing, justifying), include choosing mathematical representations and connecting them to one another and to everyday life; looking for repeated reasoning, patterns, and structure; transforming equations into other forms; and using precise vocabulary to describe complex mathematical ideas (Matsuura et al., 2013). A final framework was funneling or focusing questions (Hagenah et al., 2018), drawing observer's attention to whether the questions teachers asked funneled the students' thinking down the teacher's prescribed path to a desired outcome or focused the students' thinking on their own understanding of the concept. At the end of the session, participants collectively crafted a feedback statement for the principal of the school they were studying to share with the teachers.

Data Collection

During two interview sessions for this study, participants provided information about their professional experience and followed a video analysis protocol (see Appendix A). They observed three sets of videos of middle school classes, one in the first session and two others in the second. The first session included a video of a fifth-grade geometry lesson from the National Council of Teachers of Mathematics (NCTM) Principles to Actions website (NCTM, 2017a). The second interview session included two video clips from an eighth-grade algebra lesson from the NCTM Principles to Actions website (NCTM, 2017b) and three video clips from a sixth-grade probability lesson from the Baker Evaluation Research Consulting Group (BERC Group, 2013). At the end of each video clip, the principals were asked to describe everything they noticed about the lesson. Once they described what they noticed, principals were asked how they interpreted each event that they had discussed. After viewing all video clips from a lesson and discussing all of their observations and interpretations, the principals then described what they would highlight with the teacher in a follow-up conversation about the lesson. We used ATLAS.ti (Muhr, 2018), data analysis software for audio and video recording, to code the interviews and sort audio clips for related topics. We transcribed all salient audio clips for more intensive data analysis.

Data Analysis

We used an inductive emic approach (Tracy, 2013) to build theory about pedagogical content knowledge for leadership. Data analysis occurred in two stages. In the first stage, we used a thematic analysis (Braun & Clarke, 2023) to explore patterns in what participants attended to during mathematics lessons and how they would respond, which led to the development of the PCKL framework. In the second stage of analysis, we used qualitative content analysis methods (Schreier et al., 2020) to examine participant comments in relation to the PCKL framework and coded the level at which participants attended to classroom events or described how they would respond to teachers.

Initial Categorization: General Topics and the Presence of Mathematics

Our first pass of coding focused on what principals noticed. Nearly all of the literature that describes what principals notice focuses on content-neutral topics that could be observed in any classroom regardless of the discipline being taught. These topics often relate to classroom culture, equity, pedagogical practices, and student interactions (Humez, 2015; Johnson et al., 2011; Schoen, 2010; Weinberg, 2010). Thus, we started our coding using these four topics as well as a general topic on the discipline of mathematics. Assessment emerged as an additional common content-neutral topic during analysis. Classroom culture was later identified as related to both classroom management practices and student participation, and therefore separated into two topics. Table 2 shows the complete list of codes, with a description and sample comment from the participants for each code.

The majority of participant comments were related to pedagogical practices, leading to the creation of pedagogical practices sub-codes. Similarly, the category of mathematics was too broad to capture the variety of ways content emerged in principals' comments and was also divided into sub-codes. Tables 3 and 4 elucidate the different sub-codes for pedagogy and mathematics respectively.

Table 2: Codes Related to Principal Noticing, with Description and Example of Each.

Original Codes	Description	Example
Classroom culture: Engagement and participation	Related to how many students were actively doing work, participating in the discussion, or called on to share an idea	From a very short video, the kids are polite, but not many are participating. She did the rest of the teaching. The kiddos are kind and nice to each other, but I'm not sure how many are actively engaged in the learning.
Classroom culture: Management, rules, and routines	Related to how the classroom was managed and whether students followed routines or rules for appropriate behavior	Strong classroom expectations have been established and are followed. In terms of running a tight ship, this teacher has built a strong classroom environment.
Equity	Related to which students were asked to respond or were talked to	The two students who volunteered are the same kids who volunteered at the beginning of class. Those were two voices we've already heard from.
Pedagogical practices	Related to decisions teachers made or actions teachers took during the lesson	I'm curious about balancing the productive struggle and getting the right answer. How far do you let students go down an unproductive path before you bring them back? What is the balance the teacher is trying to strike?
Student interaction	Related to how students were grouped, how they worked together, and how they treated one another	There wasn't anyone interrupting. A kid would get to explain without anyone jumping on top of that.
Assessment	Related to whether and how teachers assessed what students understood during the lesson	When she responded to students with 'exactly right,' what about the students who didn't get it? There was a missed opportunity to assess where every student was at.
Mathematics	Related to any description where mathematics was evident	The teacher had a goal of what she wanted to walk away with — the formula for the area of a triangle. The teacher's goal superseded the importance of student growth in determining the meaning.

Table 3: Pedagogical Practices Sub-codes with a Description and Example of Each.

Pedagogical practices sub-codes	Description	Example
Interaction with students	Related to how the teacher interacted with students	She found a balance between having fun with students and moving them forward in the lesson.
Learning target	Related to the learning goal of the lesson or whether students understood the learning goal	The teacher was doing much of the talking. She was trying to help students make connections with the learning target, helping kids to move towards the learning target.
Lesson design	Related to the lesson plan	She talked about how the investigation was connected to both forms of probability. Everything was connected and she had the standards. The lesson was definitely planful.
Productive struggle	Related to whether and how students were encouraged to wrestle with the content	Students were doing the thinking and learning, not the teacher. The teacher wanted the kids to be doing the thinking, not just for getting an answer. Her questions were on the right track -- she'd add a question to have them think about it another way. They were still persisting with the same task.
Questioning	Related to how teachers questioned students and allowed for student responses, and to the types of questions asked	The teacher asked prompting questions to groups at their level. There was no right answer so it's fun for kids. What did you expect? Were you surprised by that?
Recording thoughts	Related to teacher creation of public records, or students writing down their ideas or referencing notes	The posters she was writing seemed random. I couldn't see them fully, but what was there didn't help me follow the conversation.
Task selection	Related to the tasks that students were asked to do and how the task required students to interact with content	Based on what I saw, she picked the perfect prompt. It was challenging enough to get them to think. They were talking with the teacher and with each other.
Teacher's role	Related to what a teacher does or should do	She's not just going through the motions — not just standing in front of the room giving answers. Don't lead them step by step through the work. Stop talking when you've given them enough. Let kids ask questions of themselves.
Time	Related to how time was used in the lesson, including the pacing of the lesson	I would question about the timing, giving think time, wait time, before going on to the next piece.
Use of language	Related to how precisely language was used, whether definitions were provided, and purposes for using language	Her focus on math language was appropriate for the lesson and she stretched kids who didn't want to articulate their meaning, not because of laziness. She helped kids complete their thoughts and sentences.

Some topics were addressed more frequently than others. Participants, for example, mentioned questioning more than 100 times but talked about the use of language only 24 times. Sometimes participants discussed questioning, use of language, learning targets, or tasks in a content-neutral way, and other times related these to mathematics. We coded noticing the learning target as content-neutral when a participant noted that a teacher had written a learning target on the board, because teachers in all disciplines do that. However, we coded noticing the learning target as mathematical when a participant connected the mathematics to the learning target, such as when one participant wondered whether the teacher really understood that the learning target required students to describe the area of a triangle.

Second Categorization: A Continuum of Noticing

After coding what participants noticed, we examined how participants discussed the different noticing topics. We began our analysis of how participants discussed different topics by analyzing comments coded as questioning. Participants described questioning as being a key tool that

Table 4: Mathematics Sub-codes with a Description and Example of Each.

Mathematics sub-codes	Description	Example
Learning target	Related to what the learning goal of the lesson was or whether students understood the learning goal	I don't know what the learning target was to begin with. If the learning target had something to do specifically with slopes of lines and writing equations, if the goal was the content piece, how would you know they achieved that? When will you know that?
Mathematical habits of mind	Related to teacher and student behaviors that support student learning	Students have time to be in their brains and struggle with the math and do not rely on someone else to struggle with the math for them.
Questioning	Related to how teachers questioned students and allowed for student responses, and to the types of questions asked	Clarifying questions that allowed kids to take numbers to context - this was really important. What does it mean, what does it mean for the company? Because she's pushing for the application, she's more intentionally helping kids make sense of the math.
Student understanding	Related to what students were understanding or not understanding about the mathematics or a teacher's response to student understanding	She did not use [the student]'s thinking to clean up the mistakes. The teacher didn't circle back to clarify the misconception, and what [the student] is saying doesn't make much sense either.
Task	Related to what students were asked to do and how the task enabled students to interact with mathematics	Students were able to access enough of the task to make some connections to their mathematical schema. The richness of the task warranted a deep dive into the mathematics.
Representations and tools	Related to the tools that students use to make sense of and solve problems and communicate reasoning, including different mathematical representations	The picture is a piece of supporting evidence to the algorithm. It's like students are learning the algorithm and then looking at the picture to connect it to the algorithm.
Use of mathematical language	Related to how precisely language was used, whether definitions were provided, and purposes for using language	She [the teacher] handled that interaction well. It would be easy for her to just gloss over [the student saying "point 12 cents" rather than "0.12" or "12 cents"] and say, I know what she meant. Maybe, maybe not that's what she meant, but if she tried to do the calculation as .12 cent, she is going to get a very different answer than if she used 12 cents.

teachers use for a variety of purposes, such as managing the classroom, furthering student understanding, pressing for justification, or encouraging sense-making. We were able to sort comments into four levels of noticing, which we summarize in Table 5. Participants interpreted questioning as serving classroom management and assessment purposes when they noticed teachers asked questions to prompt student engagement, provide formative assessment information, or keep the class on pace. We categorized such purposes as general pedagogy because teachers in all disciplines use questioning for these purposes. In general pedagogy, observers attend to the frequency and depth of questions teachers ask and to whom.

At times, participants noticed that questioning connected instructional decisions with student learning of content, but the particular content did not matter. When participants noticed that teachers drew right answers out of students or used questioning to fix a student's misconception, content was part of the observation, but observers could notice the same purposes for questioning in a science, history, or art class. Because these observations relate pedagogical decisions to content but are not discipline-specific, we labeled these quotations as *parallel content and pedagogy*. At this level, the observer attends to teacher questions and student responses.

A third approach to noticing focused on an intersection of content and pedagogical practices. Participants regularly described questioning as a means of encouraging students to move beyond right answers and to explain their thinking. Mathematics or mathematical topics were rarely explicitly mentioned when discussing questioning. However, when participants noticed that questions pushed students for further explanation, we interpreted this as demonstrating understanding that student explanation of reasoning is key to mathematical learning. We labeled these comments as an *intersection of content and pedagogy*. At the intersection of content and pedagogy, observers attend to whether the teachers' questions push students to explain their thinking.

Finally, participants noticed when teachers used questioning to push students to engage with the content as emerging mathematicians. Principals who noticed that the teacher questioned students to make sense of an equation in context, to make connections between different representations, or to generalize mathematical principles demonstrated an understanding that questions can press for the use of mathematical Discourses. We labeled these comments as *pedagogy and mathematical Discourses*. At this level, observers attended to whether questions pushed for student enactment of the SMPs such as sensemaking, modeling with mathematics, using structure, or connecting representations. Table 5 provides sample purposes and related quotations that participants gave for questioning at different levels.

Noting that participants could interpret purposes for questioning along a continuum, we then considered how participants noticed *representations and tools*. How participants described the purpose for classroom use of *representations and tools* fell along a similar continuum as questioning. We categorized observations related to using tools to support student engagement as part of "just good teaching," or *general pedagogy*. At this level, participants described manipulatives as answer-getting devices, engaging entry points into the task, or a way to assess student understanding. At the *parallel content and pedagogy* level, participants noticed that teachers used representations and tools to demonstrate key ideas. As with questioning, at this level, content was mentioned in relation to instructional decisions, but not in a disciplinary-specific way. At the *intersection of content and pedagogy* level, participants noticed how multiple representations supported students in developing an understanding of important concepts or visualizing mathematical relationships. Finally, participants interpreted the use of representations and tools as a means of providing opportunities for students to behave as emerging mathematicians by solving complex problems with different representations or generalizing mathematical principles at the *pedagogy and mathematical Discourses* level. See Appendix B for examples that show how principal noticing related to *representations and tools* fell along the same continuum we found with *questioning*.

We found that participant noticing of all seven mathematical topics followed the same pattern as *questioning* and *representations and tools*. Appendix C provides comments at different levels for two additional mathematics topics: *use of language and learning target*.

Table 5: Purposes of Questioning and Examples Coded at Different Levels.

Level of noticing	Purpose	Example
General pedagogy	Equitable engagement	There was not a diversity of students she was asking questions to. In the class of about 25, one girl spoke 3 times.
	Draw students into the conversation	Then the teacher asked another student, 'How do I write this?' Why did she ask that question. I think she was just trying to get someone else talking other than herself.
Parallel content and pedagogy	Provide hints to fix student understanding	When the student gave the answer, and her response was, 'Is it just a one?,' she was just giving them the answer that you did something wrong, fix it.
	Funnel student thinking	I noticed that the teacher had a hope for the students' activity and what they would conclude. She asked very leading questions as opposed to more open-ended questions.
Intersection of content and pedagogy	Encourage students to move beyond right answers	She's moving beyond right answers and into the thinking behind right answers. She's patient, giving kids time to explain their thinking.
	Describe why the answer is right or wrong	She said, 'Whether it works or not, tell me why," because we can learn from wrong answers as much as right answers.
Pedagogy and mathematical discourses	Make sense of problems	The teacher is asking questions like, 'Why is the equation working this way? Why does it end up telling us what it tells us?' The conversation is helping the students put the pieces of the puzzle together.
	Make use of structure	The questions were probing, pushing kids down a path of inquiry. She wants them to figure out the difference between [1/2 the (length times width)] and [(1/2 the length) times the width].

Results

Development of the PCKL Framework

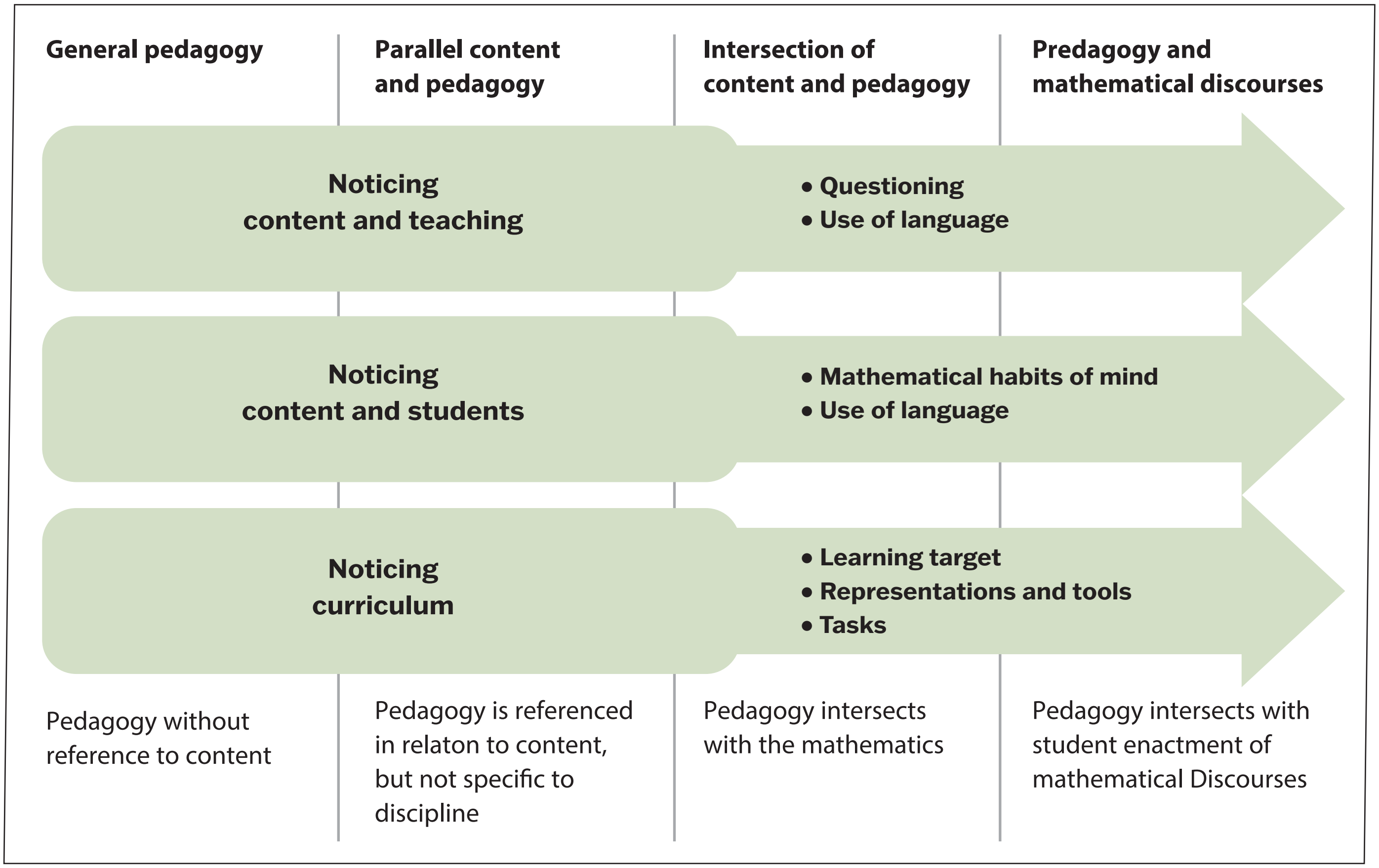
Observations for each mathematics topic fell along the continuum: general pedagogy, parallel content and pedagogy, intersection of content and pedagogy, and pedagogy and mathematical Discourses. The PCKL framework describes the different levels along the continuum for each topic. Figure 2 (see pg. 42) provides a visual overview of the PCKL framework which is presented with descriptions in Tables 6 - 8 (see pgs. 43-44).

Similar to van Es' framework for learning to notice student thinking (van Es, 2011), the PCKL framework considers *what* principals notice, as shown in the arrows, and *how* they notice, as shown in the columns. *What* they notice are aspects of the three categories of PCK (Hill et al., 2008) — knowledge of content and teaching, knowledge of content and students and knowledge of curriculum. *How* they notice builds along the continuum from general pedagogy towards an understanding of mathematical Discourses (Gee, 2015b; Hauk et al., 2014).

Leader Noticing of Mathematics

After building the PCKL framework, we returned to each of the participants' observations, interpretations, and responses, coding each for the level of noticing based on the framework. Because every comment had been coded as content-neutral or mathematical, the content-neutral comments were necessarily level 1. The other comments were coded as level 2, 3, or 4 based on the framework. The first two authors met regularly throughout the coding process to maintain clear definitions of each set of the noticing levels. The authors also used specific examples to monitor coding rules and adjust as needed. This analysis allowed us to compare the level of attention and response, providing insight

FIGURE 2.
Levels and Topics of Noticing in the PCKL Framework



into the connection between what participants noticed and what they would discuss with their teachers. Table 9 shows the highest level of attention (Att) and the highest level of response (Res) about each topic for each participant. The first group of participants had PD for leaders of mathematics, the second group taught mathematics, the third had PD for mathematics teachers, and the fourth had neither taught mathematics nor had mathematics-specific PD. Level 0 on the table indicates that participants did not address that topic during their interviews, and levels 1 – 4 represent the levels on the PCKL framework.

The data from Table 9 (see pg. 45) indicate that, just as novice teachers can be taught to observe important events like more experienced ones (Roth McDuffie et al., 2013), principals can learn to notice mathematical Discourses with intentional PD. Only the participants who had PD targeted at leaders of mathematics discussed several topics at level 4. All three noticed *habits of mind* and *representations and tools* at a high level, and Blake and Henry noticed *questioning*, *learning targets*, and *tasks* at level 4. During their interviews, all three of these participants specifically acknowledged the PD experiences where they learned about how high-quality instructional practices influence student learning. For example, Henry said,

> Through the [professional development], I really grew to understand how you ask a question, how you give students time to really process that, and how when answering the question, you allow for multiple pathways to get to that answer, that you have students show their work, explain, and describe why they do it, and not simply respond with, "Yes that's correct," and "No, that's incorrect."

This participant alluded to focusing on *questions* and student *habits of mind* as well as *tasks* that have multiple pathways to find solutions as a result of the PD experience. The learning experience helped him to understand how pedagogy is interconnected with student enactment of mathematical Discourses.

Table 6: Pedagogical Content Knowledge for Leadership (PCKL) Framework for Noticing Content and Teaching.

Topics	**Level 1 General pedagogy**	**Level 2 Parallel content and pedagogy**	**Level 3 Intersection of content and pedagogy**	**Level 4 Pedagogy and mathematical Discourses**
Questioning	Attends to the frequency and depth of questions teachers ask and to whom	Attends to teacher questions and to student responses	Attends to whether the teachers' questions push students to explain their ideas	Attends to whether questions push for student enactment of the Standards for Mathematical Practice (such as sensemaking, modeling with mathematics, and using structure or multiple representations)
Use of language	Attends to who is doing the talking and how students and teachers talk with one another	Attends to whether students and teachers use mathematical language Interprets focus on acquisition of vocabulary and definitions as a priority of mathematics instruction	Attends to teacher press for use of precise mathematical language Interprets precision of student language as an important mathematical learning outcome	Attends to student use of precise language to explain mathematical ideas Interprets the precision of mathematical language as an important tool for articulating reasoning

Table 7: Pedagogical Content Knowledge for Leadership (PCKL) Framework for Noticing Content and Students

Topics	**Level 1 General pedagogy**	**Level 2 Parallel content and pedagogy**	**Level 3 Intersection of content and pedagogy**	**Level 4 Pedagogy and mathematical Discourses**
Mathematical habits of mind	Attends to how the teacher interacts with students about learning: e.g., teacher asks questions or tells answers, allows students time to think on their own, provides immediate feedback, and praises students	Attends to student responses to teacher interactions: e.g., student demonstrates frustration or confusion about what teacher expects, feels empowered to support teammates, moves into productive or unproductive struggle	Attends to the role of the teacher in supporting student learning: e.g., teacher asks questions about how ideas are connected, pushes for higher levels of thinking and analysis, requires justification of ideas with evidence, prompts for metacognition, and withholds evaluation of student solutions	Attends to student behaviors that support learning: e.g., student explains thinking, makes hypotheses, justifies, generalizes, and articulates answers, uses math vocabulary, provides evidence to support reasoning, tries and abandons different ideas
Student understanding of mathematics	Attends to how many students appear engaged and which students speak	Attends to whether the class can explain the mathematics	Attends to individual students' mathematical thinking and explanations	Attends to the individual students' mathematical thinking and the connections between teaching strategies and student mathematical thinking, justification, generalizations of mathematical principles

Table 8: Pedagogical Content Knowledge for Leadership (PCKL) Framework for Noticing Curriculum.

Topics	Level 1 General pedagogy	Level 2 Parallel content and pedagogy	Level 3 Intersection of content and pedagogy	Level 4 Pedagogy and mathematical Discourses
Learning target	Attends to whether the teacher has a clearly articulated learning target for students and whether the teacher assesses for student proficiency of the learning target	Attends to alignment between the learning target and the assigned task or grade level	Attends to how the teacher's decisions led to student understanding of the learning target	Attends to how the Standards for Mathematical Practice (such as inquiry, justification, and generalization) are embedded in the learning goals
Representations and tools	Interprets the use of representations and tools as a means for students to find answers, as an entry point for engagement, or as a means of assessing student understanding	Interprets the use of representations and tools as a means of showing mathematical concepts to students	Interprets the use of representations or tools as a means of developing conceptual understanding or visualizing mathematical relationships	Interprets the use of representations or tools as a means of solving complex problems or generalizing mathematical principles
Tasks	Attends to whether the task engages students	Attends to whether students follow a prescribed pathway or can access the task through multiple solution pathways	Attends to whether the task encourages students to reason about mathematics and show connections between different representations	Attends to whether the task requires justification or generalizing of mathematical principles or incorporating the use of different representations including context

On the other hand, mathematics teaching experience did not appear to be as influential in developing this same lens. Matthew and Peter, who taught secondary mathematics for 17 and 12 years, respectively, noticed only one topic each at level 4. Like those with leadership PD, Matthew and Peter recognized *representations and tools* as important to learning math, as did Sam who had PD for teachers, but none described them as being interconnected with mathematical Discourses as Blake and Henry did. For example, Peter and Matthew described the visual representation as a means of understanding the mathematical concept and connecting an image to the rule. Peter said, "I want her to be able to talk about what individual students understood and what she would do to help students who aren't yet understanding. Students would be clear about the model and why it works and connect it with algorithmic language." Neither described the use of tools as a means of behaving like a mathematician. Blake, however, described representations and tools as important for making sense and justification, a means of acting like an emerging mathematician, when she said, "This [task] was a way to look at math. This was a way to look at shapes. This was a way to fold and count, and models for this unit are important as a way to justify their thinking." When describing that seeing how graphs and equations related to a company in context is really important, she added, "We keep the mathematics if it has meaning and we've made a connection to it." Unlike their peers, one former mathematics teacher never demonstrated level 4 noticing, however.

The data also show that principals who attend to mathematical Discourses are positioned to direct teacher attention to them during follow-up conversations. In one case a participant demonstrated how what she noticed during the lesson prompted her to respond to the teacher at level 4. Of her observation, Blake said,

> I ended up recording her questioning prompts. 'Where are you coming up with this?' 'Where did you get the ½?' 'Why?' 'I'm curious.' 'Is there another way?' 'Is that

Table 9: Levels at Which Participants Attended (Att) and Responded (Res) to Mathematical Topics.

Participants grouped by experience (PD or math teaching)	Question		Language		Habits of mind		Student under-standing		Learning target		Reps & tools		Tasks	
	Att	Res	Att	Res	Att	Res	Att	Res	Att	Res	Att	Res	Att	Res
Blake	4	2	2	0	4	4	3	0	4	4	4	1	4	0
Bradley	0	1	4	0	4	4	0	0	0	2	3	0	4	0
Henry	4	1	3	0	4	0	4	0	4	2	4	0	0	0
Matthew	4	4	3	0	3	3	0	0	1	2	3	0	2	0
Peter	3	1	0	0	4	4	2	0	2	2	3	3	0	1
Stuart	2	1	0	0	2	0	2	0	1	1	3	0	3	3
Sam	2	0	2	0	3	0	1	0	4	1	3	0	3	0
Lindsey	3	0	3	2	3	1	2	0	2	0	3	0	2	0
Warren	0	2	1	0	2	0	0	0	2	1	1	0	3	0

Note: This table shows the highest level that each participant attended to each topic (Att) and the highest level that each participant said they would respond to the teacher (Res). The first group had PD for mathematics leaders, the second group taught mathematics, the third had PD for mathematics teachers, and the final group had no mathematics PD nor math teaching experience.

> the same thing?' 'How do you know?' Clearly, she is looking for student knowledge, she's not leading them anywhere, she's not saying, 'Oh you're on the right track.' 'I appreciate your answer.' 'Let's build on that one.' It was very much an open-ended, 'What could this look like?' 'What is your mathematical thinking?' It wasn't until the very end that she even put numbers in there to check it. Her purpose was for students to just explain their thinking. 'How did you fold the paper?' 'How did you count?' 'Where did these numbers come from?' 'How did you make a formula?,' no matter how big the right triangle ended up being.

In this observation, the participant noticed questioning strategies that lead to mathematically productive habits of mind and using representations to make sense of the mathematical concepts. In describing how she would follow up with this teacher, she said she would address the teacher's questions with a level 4 understanding.

> Her question, 'Does this apply to all right triangles?' would be part of my follow up in terms of what does this mean for the rest of this unit. 'What other shapes have you done? Where are you? Where is this going? Do you have other ways to model this with different shapes? How are you using this formula?' Just, 'Where is this going?' because that's how she's leaving it. And how can kids generalize that math information?

This planned response to the teacher demonstrated an understanding that the mathematical representations she attended to in her observation were essential to generalizing mathematical relationships, not just for the right triangle that students were working on but for other shapes as well. The principal's reinforcement of that important idea during the follow-up would focus the teacher's reflection on how her pedagogy could support student enactment of mathematical Discourses.

Although it appeared that noticing mathematical Discourses enabled participants to direct teacher attention to the Discourses, it did not appear sufficient. Often when participants noticed mathematical Discourses being enacted, they would respond about content-neutral or parallel content and pedagogy. Bradley observed mathematical Discourses during the lesson:

> Most of [the students] were explaining their thoughts. The one little gal was explaining how to punch it into her calculator – the task was the opposite direction – but she was understanding how to manipulate the table… It was all the right stuff. They obviously had enough to mess around with, they were making hypotheses or postulates, saying, "Here's my answer," they were forced to justify that, they were forced to collaborate with their peers. And then she would also push by asking, "Is that the only way?" "Is there another equation that's there?" She still hasn't given them the answer. She let them mess with it for a little while.

When asked how he would respond to the teacher, he said he would ask how she would ensure that every student understood the mathematics because often with an inquiry-based approach, the teacher may be unaware of the students who do not understand. Even though the participant noticed the importance of conjecturing, generalizing, justifying, and manipulating different representations, he dropped to a level 2 response related to assessment of the content of the lesson much as he might in any other classroom. While no one would disagree that we want teachers to assess student learning, this participant missed the opportunity to also prompt further teacher reflection on the interplay between pedagogy and mathematical Discourse.

In no case did participants in this study indicate that they would ask a teacher to reflect on mathematical Discourses at a level 3 or 4 on the framework unless they had addressed the topic at that level when describing it during the attending phase of the protocol. Thus, data from this study indicates that attention to mathematical Discourses may be necessary but not sufficient for principals to include them in their responses.

Where those who had PD for leaders were more likely to attend to mathematical Discourses, the former mathematics teachers, Matthew, Peter, and Stuart, showed more willingness to respond at levels 3 or 4 on topics they had observed at that level. Where Matthew noticed 4 topics at level 3 or 4, he responded at level 4 twice; Peter responded at level 3 or 4 in two of the three topics he noticed at that level; and Stuart responded at level 3 on the only topic he noticed at level 3. Those with PD for leadership had a lower rate of response at higher levels, even when they noticed higher levels on more topics. Blake, who noticed at level 3 or 4 in six topics, only responded at level 3 or 4 on two of them; Bradley responded at level 4 on only one of the four topics he noticed at a higher level; and Henry never responded above level 2 even after noticing at levels 3 or 4 in six topics. Matthew explained why he would address the mathematics while others might shy away from it, stating, "I would focus on the math since I'm a math teacher. It's easier to focus on the math." He added that he may not discuss the content with a Language Arts teacher.

Henry, who regularly noticed mathematical Discourses, said he would address content-neutral topics in follow up conversations. After noticing two key inflection moments in the class where he wished the teacher would stop and question the students for deeper understanding, he reflected on how he would respond with, "I don't know if I'd discuss questioning and discussion or student engagement. Engagement is an easy one because teachers know when kids are paying attention or not. In each section, who's engaged in discussion? How can you tell?" Even armed with a clear understanding of key mathematical moments and related pedagogical opportunities, he dropped to discussing the student engagement with the teacher. He added an explanation of why he would raise content-neutral topics with teachers,

> I know what instructional practices look like across content areas. I know what questioning, discussion, and engaged learners look like. I do not necessarily know the math piece, so I'd use Kathy Norwood's approach of drawing the ideas out of the teacher.

Henry noticed five topics at level 4: *questioning, habits of mind, student understanding, learning targets*, and *representations and tools*. However, when asked how he would respond, he said he would ask teachers to reflect at lower levels related to student engagement or assessment. His attention to the mathematical Discourses was not sufficient for him to focus teacher attention on key mathematical moments.

Some evidence in this study, therefore, suggests that previous mathematics teaching experience may aid principals in providing feedback to teachers when they attend to key mathematical events. Others may need support building confidence and strategies for how to respond about pedagogical decisions that promote mathematical Discourses.

Discussion

Evaluation and supervision systems rest in an unstated assumption that principals can support teacher learning without the knowledge of discipline-specific instructional strategies or, in the case of mathematics, an understanding of mathematical Discourses. As Goodwin (1994) found, communities of practice socially construct shared ways of seeing. Teachers stand at the intersection of different communities of practice that have complementary but not fully aligned professional visions. Effective teachers rely on a shared vision of good instruction that can include such skills as focusing intentionally on content standards, choosing tasks that engage students, or asking questions that further student understanding, a vision that is clearly articulated in rubrics for teacher supervision. However, they must also rely on pedagogical content knowledge to push students to enact the practices that promote content-specific learning. The professional vision of mathematics teachers, therefore, must include more than a vision of "just good teaching" and more than just PCK. A mathematics teacher's professional vision must include a vision for what mathematical Discourses look and sound like in classrooms and what teachers do to elicit them.

During evaluation processes, principals observe classrooms to gather evidence that documents the effectiveness of instruction and work with teachers to compare that evidence to rubrics. The evaluation process typically uses gathered data, the rubric, and conversation to determine the teacher's strengths and areas for growth. Because principals frequently gather the data that are discussed, what they notice during the lesson shapes the conversation. If their data does not contain discipline-specific events, some key features of lessons may pass without critique and salient opportunities for improvement may be missed. As Bradley explained,

> What matters is the process that [students] took to get to that [answer] and then what their thinking is. And then having another student be able to come by and say, 'Yeah, I got to this answer which may or may not be the same, and I got to it in a completely different way using a completely different model.' And so, the idea of open questions and really allowing students to explore their own thinking and make that explicit in the classroom [matters].

The components of instruction that matter for teachers who are learning to enact high-quality instructional practices and for principals who supervise them include the big D discourses and the important mathematics that is embedded in them. Bradley's comment indicates an awareness of the power of mathematical Discourse that he developed during leadership PD. Principals who can support a classroom teacher in developing such a vision of effective instruction centered on student enactment of mathematical Discourses are in a strong position to support powerful mathematics instruction throughout their schools.

Experienced principals who have used instructional frameworks for teacher evaluation and supervision are well-versed in effective general pedagogy. The PCKL framework can further advance principals' abilities to notice important classroom events through content-specific awareness. An observer at level 1 on tasks "attends to whether the task engages students." At level 4, an observer "attends to whether the task requires justification or generalizing of mathematical principles or incorporating the use of different representations including context." Content-neutral pedagogical observations thus form a strong foundation that can be built upon for observations about mathematical Discourses. An observer who already attends to the assigned task can learn to attend to the mathematical characteristics of the task. Rather than asking principals to abandon what they know, the PCKL framework demonstrates how principals' current knowledge of general pedagogy is a valuable asset they can build upon to support the mathematics teachers they supervise.

Mirroring what researchers have found about teacher noticing (Sherin & van Es, 2009), this study provides evidence that principals can learn to attend to mathematical Discourses necessary to support teachers in strengthening their own vision of the Discourses. Professional development providers can support mathematics leaders' growth by focusing their attention on how it looks and sounds when students talk and behave as mathematicians. PD experiences for principals could draw attention to some of the mathematical topics found in the PCKL framework, such as characteristics of rich mathematics tasks, student use of mathematical tools and representations, or focused questioning to support learning. This study suggests that PD opportunities with an intentional focus on level 4 of the PCKL framework, particularly on student enactment of mathematical Discourses, may help principals attend to key mathematical events during classroom observations. School leaders who are learning to notice the mathematical Discourses at a high level will likely need further support to learn to focus a teacher's lens on significant mathematical events.

Using the PCKL Framework

The PCKL framework can guide the work of PD providers by articulating how mathematical topics are directly linked to the interconnected relationship of pedagogy and mathematical Discourses. For example, PD for leaders might highlight mathematical Discourses by including a video case (Johnson & Mawyer, 2019) that allows principals to observe students justifying their reasoning about a concept they generalized when doing a rich mathematics task. A reflective conversation about that video case could focus on both the construction of the mathematics task to promote Discourses and student behaviors that foster mathematical habits of mind. Similarly, a team of leaders might do a learning walk (Elmore et al., 2009; Fisher & Frey, 2014) in mathematics classrooms to observe how Standards for Mathematical Practice are embedded in the learning targets. During these PD experiences, facilitators could support principals in learning to respond to teachers by asking them what they would focus on during a post-observation conversation or collaboratively creating questions they would ask teachers based on what they observed. Using the PCKL framework as a guide in planning learning experiences for principals may support PD providers in furthering principal attention to important events.

Mathematical content knowledge may not be essential for learning to attend to mathematical Discourses, but principals who lack the mathematical background of former teachers may need additional support to develop strategies for responding. This finding is apparent from Henry who had a clear vision of mathematically productive mathematics classrooms but said he would use lower-level responses in follow-up conversations with teachers. As principals learn to attend to mathematical Discourses, PD providers may also consider how to intentionally support those who have not taught mathematics so they learn to respond to teachers about key mathematical moments when they notice them.

Even if they do not have access to external PD providers, principals can use the PCKL framework in their own work with teachers. If a principal feels particularly confident with a content-neutral topic or regularly observes parallel content and pedagogy, the framework can provide guidance for furthering what they look for in mathematics classrooms. For example, during observations, principals who regularly pay particular attention to who speaks and for what purpose can also listen for how those students use mathematical vocabulary. Is learning vocabulary treated as an important learning target, as an essential tool for making sense of the mathematics, or as necessary to effectively justify or discuss a generalization of key mathematical concepts? Once principals articulate for themselves how language is used, they may consider how to provide teacher feedback that further develops student mathematicians and supports teachers in becoming better than "just good teachers."

Principals can also enlist their teachers in developing a shared vision of a classroom alive with emerging mathematicians. The mathematics team and their supervisor could form a video club to watch lessons curated by the mathematics research community and negotiate what enacted Discourses look and sound like. They may choose just one topic and study it using multiple videos or watch the same video multiple times, changing the observation focus. A principal could also take mathematics teachers through classrooms and discuss when they observe student enactment of Discourses. Together, they could consider how teachers would like to receive feedback focused on higher levels of the PCKL framework.

Conclusion

The PCKL framework presented in this paper shows a progression of noticing along a continuum from general pedagogy to student enactment of mathematical Discourses in the three components of PCK (Hauk et al., 2014). We used the framework in this study to articulate levels of principal noticing. In so doing, we found that principals were able to learn to attend to key mathematical classroom events and that they may need additional support to know how to respond to teachers. We contend that the PCKL framework can provide much-needed guidance to PD providers for supervisors of mathematics teachers or to principals who strive to notice more during lessons.

The CCSS-M charges teachers with developing mathematically proficient students, and the rubrics in teacher evaluation documents require that principals support teachers in developing the necessary pedagogical content knowledge. This study identifies what principals need to know about the discipline of mathematics if they are to meet this challenge, and suggests ways that they might learn about mathematical Discourses in their work with teachers. Learning opportunities that move principals along the PCKL continuum of noticing may position them to play a key role in fostering powerful mathematics teaching and learning.

Because this study built the PCKL framework from what the participants noticed, the findings may be limited. There may exist other important elements of PCKL that this set of participants did not name. The videos were designed for use with teachers and therefore leaned towards teacher behaviors, so using videos focused solely on students may have drawn out other important ideas. Replicating this study with participants who have expertise in mathematics education may also add to topics in PCKL. The number of participants in the study is also a limitation in drawing generalizable conclusions about the importance of PD for supervisors of mathematics. Although the evidence in this study mirrors teacher noticing research indicating that PD focused on noticing at high levels appears necessary but not sufficient to know how to respond well, the number of participants and the relatively small data set indicate that further research would be beneficial.

This study does, however, lay the foundation for articulating the mathematical content knowledge that exists at the intersection of content, pedagogy, and leadership. As the province of school leaders who focus on improving student learning of mathematics, the PCKL framework narrows the scope of their knowledge base and focuses their lens on what really matters – the emerging mathematicians at their school and a culture of learning that supports their development. ✪

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Appendix A
Video Noticing Protocol

We are going to watch three video clips of teachers in math classrooms and I will ask you to record anything you notice as we watch. We'll then discuss what you noticed and your interpretation of the event. After we've discussed everything you noticed, I'll ask you how you would respond to the teacher as though she were on your staff, in her evaluation or in a follow-up conversation, or perhaps in how you might consider your work with your entire staff.

The first video is in a 5th classroom discussing the formula for area of a triangle. This teacher is working to support student engagement in and understanding of mathematics and to develop procedural fluency through conceptual development. The students have worked on a task and we enter as they are debriefing the task.

We'll watch the two videos of an 8th grade classroom in sequence. This teacher is working on supporting student engagement in and understanding of mathematics and posing intentional questions. The first video is taken during the work time of a lesson, and the second is a whole class discussion of what was learned during the lesson. I will ask you to tell me what you noticed and how you interpreted what you noticed about the first video before starting the second, but I won't ask you how you would respond to the teacher until after we've discussed what you noticed in the second video.

The final set of three videos is from a 6th grade classroom. The teacher is teaching about probability. The first video is the opening of the lesson and is a whole class discussion. The second video is taken during student exploration time and focuses on the teacher's interactions with small groups. The third video is the debrief of the lesson.

a. We'll watch the video of the lesson. As we watch, record anything you notice. You'll be able to watch the video or portions of the video as many times as you'd like. If there's a portion you would like to watch again, please record the time and we can go back to that part of the video.

b. (After first observation) Are there any parts you want to watch again or even the whole video?

c. Would you please share what you recorded? In what you noticed, who was involved and what were they doing?

d. How do you interpret what you recorded?

e. (After completing steps 1 a - d for all video clips of the same lesson). What norms appear to be present in the classroom we just observed?

f. How would you respond to this teacher if you were her supervisor, either in her evaluation or a follow-up conversation? How might you use what you learned from this lesson in your broader work with your staff?

(If not already discussed, follow up with) What did you notice about the mathematics that the students were engaged in? What did you notice about the way students were thinking about the mathematics?

Appendix B
Ways of Observing Representations and Tools with Examples

Level of noticing	Purpose	Example
General pedagogy	Engage students	The kids were rolling dice and they seemed to be having a good time. I mean, rolling dice is engaging.
	Help students find right answers	Every kid came to same conclusion with the manipulatives. There was likely a prescribed way to use them since they all got the same results.
Parallel content and pedagogy	Demonstrate a disciplinary concept to students	The teacher was trying to formulate a picture for students so they can see. It seems like the teacher was showing them.
Intersection of content and pedagogy	Support conceptual understanding	She asked the student to explain the equation, where the equation came from. If they were provided the equation then what does it mean? [Students need to understand] how their tool works, understand how it represents what's happening.
	Visualize mathematical understanding	The teacher has students make a visual representation of their thinking. They were able to use the manipulatives to show what they were thinking.
Pedagogy and mathematical Discourses	Generalize mathematical principles	What students would normally understand without the model is that [1/2(l·w)] is the same thing as [(1/2 l)·w]. But they're not the same thing. They are two different things that yield the same answer because they represent two different [ways of visualizing how the formula is constructed]. Without the model, I think that's really hard to visualize but when the students cut and then flipped them, then you can see why [the equations look different].
	Solve complex problems using different mathematical representations	They have created the algorithm and what they think that means, they've graphed it and they're all in the same spot, so they should be able to tell you what is going to happen in any of those equations [that intersect at the same point], which is what they all have in common within the context.

Appendix C
Examples of Comments at Different Levels

Use of Language

Level 1: General Pedagogy: "The kids were engaged and very on target with their language."

Level 2: Parallel Content & Pedagogy: "They are learning productive language from the beginning rather than saying, 'That number or that number.' They are learning the proper language from the start which will help them as they grow and mature, those pieces will be with them already. The teacher is doing a good job of teaching those basic expectations of teaching those labels of what things are."

Level 3: Intersection of Content & Pedagogy: "She consistently would ask the kids to dig a little bit deeper for that understanding and explanation piece to try to take the thoughts and use the correct vocabulary to say it out loud."

Level 4: Pedagogy & Mathematical Discourses: "Based on where you put those in your calculator, what are those things? The student's being forced to explain so it's not just what are those things really so it's not just what you punch in next. Explaining your thinking is, 'Why did you pick that?' That vocabulary to be able to explain the depth of understanding, the 'Why did you think that would work?'"

Learning Target

Level 1: General Pedagogy: "I want to know what her hunch is about the students' ability to demonstrate proficiency with the learning target for today."

Level 2: Parallel Content & Pedagogy: "I don't know whether learning about right triangles is a fifth-grade standard. That would be something I would have to look into before going in to observe a teacher."

Level 3: Intersection of Content & Pedagogy: "The teacher was doing a lot of the talking because she was trying to help students make connections with the learning target. She was moving students towards the learning target."

Level 4: Pedagogy & Mathematical Discourses: "The goal on this task is not the formula. If that was the case, you would just give it to them and write it down. Sometimes when teachers don't understand the distinction, they'll do this kind of lesson plan but they will not have that kind of patience. The goal is to understand the formula, to understand where that formula comes from."