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Using Scenario Tasks to Elicit Teachers' Algebraic Thinking: *A Recommendation for Professional Development*

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One of the greatest challenges for providers of professional development in mathematics is to determine the degree to which professional development experiences help teachers to improve their mathematical content knowledge and the pedagogical strategies they employ. Collecting such information may not only serve to evaluate the effectiveness of the professional development opportunities offered, but it may also inform the design and content of subsequent professional development sessions. Traditional paper-and-pencil mathematics content tests may seem to be efficient in assessing content knowledge, but they are limited in that they create a degree of anxiety among teachers and are viewed to be threatening. Furthermore, such measures test mathematical content knowledge often at the exclusion of pedagogical content knowledge. Other approaches may include personal interviews or classroom observations, but these, too, may be limited for similar reasons.

One way to elicit mathematical understandings and pedagogical strategies is to present teachers with a realistic classroom scenario in which student responses are plausible but problematic, and ask teachers how they would respond with respect to the correctness of students' ideas (i.e., elicit mathematical content knowledge) and how they would approach or resolve the conflicts or dilemmas (i.e., elicit pedagogical strategies).

In the course of conducting classroom observations for a National Science Foundation-funded Local Systemic Change Project[1], we began to collect "teachable moments" — capsule instances where an unexpected student response paved the way for a significant mathematical insight if further pursued. For a variety of reasons (e.g., lack of time, lack of confidence to follow a student's lead, lack of content knowledge, or lack of interest in the student's response), some teachers chose not to address the issues raised by these unanticipated responses. With a goal toward analyzing middle school teachers' algebraic thinking and the pedagogical strategies they employ, and to understand more fully why some teachers chose not to pursue their students' reasoning, we developed scenarios (Scher, Curcio, & Weinberg, 2004) based on these actual classroom observations as well as from *Connected Mathematics* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), the adopted curriculum.

Because the development of algebraic reasoning is a critical component of the middle school mathematics curriculum, in this paper we present three algebra-related scenario tasks that may be useful in eliciting teachers' thinking related to algebra as well as their instructional strategies. As noted earlier, these scenarios are based on actual classroom situations. Accordingly, as providers of professional development elicit and analyze teachers' responses to the

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scenarios, they may use the information to structure and design future professional development sessions.

THREE SCENARIO TASKS

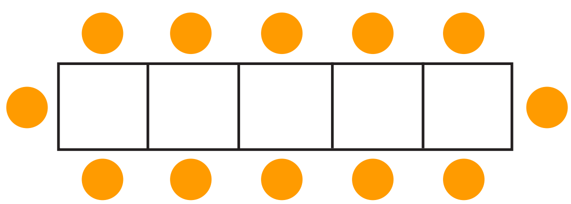
Scenario Task A: Seating Capacity

This task, observed in a grade 7 classroom, is a variation of a problem in *Covering and Surrounding* (Lappan et al. 1998b, p. 32).

SCENARIO TASK A: SEATING CAPACITY

Students in your 7th-grade algebra class are working in groups to answer the following question:

A square table can seat four people, one on each side. When 5 square tables are placed side by side, as shown below, 12 people can sit around them—5 on each side and 2 on the ends. How many people can sit around n square tables when they are placed side by side?



One group says: "n people can sit on each of the two long sides, and two people sit on the ends. So the total number of people is 2n +2."

Another group says: "If there's just one table, then 4 people can sit. Each time we add a table, that increases the number of people by 2. Thus the total number of people is 4 + 2n."

How would you help the groups in analyzing these two responses?

In particular, where is the error in the above work and how can it be corrected?

It should be noted that when the class came together to review the results of the small group work, the teacher was faced with two seemingly plausible lines of reasoning. Because one led to an incorrect algebraic answer, 4 + 2*n*, the teacher dismissed it without considering the merits of its underlying reasoning, and how it could be amended. Could other teachers do better with this "teachable moment?" Scenario Task A is designed to find out.

Scenario Task B: Perimeter versus Area

In this lesson, we observed a grade 6 discussion of the relationship between the perimeter and area of a square. Scenario Task B is based on the response of one student who noticed an unexpected numerical pattern in the data. Is this pattern a mere curiosity or can it be related to algebraic thinking? Finding and articulating the algebraic connection is the object of this task.

SCENARIO TASK B: PERIMETER VS. AREA

Students in your 6th-grade algebra class are creating a table that lists the perimeter and area for squares of varying sidelengths:

Sidelength of square	Perimeter	Area
1	4	1
2	8	4
3	12	9
4	16	16
5	20	25
6	24	36
7	28	49

A student notices an interesting pattern in the table that she shares with the class:

A square with side length 5 and perimeter 20 has area 5 x 1 + 20 = 25.

A square with side length 6 and perimeter 24 has area 6 x 2 + 24 = 36.

Extending this pattern across the table, she finds:

Sidelength of square	Perimeter	Area
1 x −3	+ 4	= 1
2 x −2	+ 8	= 4
3 x −1	+ 12	= 9
4 x 0	+ 16	= 16
5 x 1	+ 20	= 25
6 x 2	+ 24	= 36
7 x 3	+ 28	= 49

How would you proceed with your class from here? Explain why this numerical relationship exists.

Scenario Task C: Binomial Expansion

In this lesson, we observed a discussion on binomial expansion. Although this topic was not part of the adopted NSF curriculum, the teacher of this advanced grade 8 class had chosen to include it in her course.

SCENARIO TASK C: BINOMIAL EXPANSION

Using algebra, you show your 8th-grade algebra class why $(a + b)^2 = a^2 + 2ab + b^2$*. On a test, however, many students write:* $(a + b)^2 = a^2 + b^2$*. How might you help your students to understand this identity?*

Please write a response that is detailed enough to allow another teacher to follow your ideas and use them as a basis for a lesson in his or her own class.

Categorizing Responses to the Three Tasks

We administered the three scenario tasks to approximately fifty mathematics teachers and mathematics coaches in a local New York City community school district, and graduate students in secondary mathematics education at a local university. We conducted this exploratory project to examine the degree to which responses varied and how the responses might be used to reveal teachers' algebraic thinking. We found that not only did responses to each scenario vary considerably from one another, but that they had distinguishing characteristics that revealed teachers' approaches that emphasized numerical examples (Response Type 1), using a table (Response Type 2), developing a generalization (Response Type 3), or making connections or extending the solution (Response Type 4).

All responses per scenario were read independently by each of the three authors of this article and classified into one of the four categories (i.e., Response Types 1, 2, 3, or 4). In creating these categories, we were guided by the belief that regardless of the quality of the curriculum materials or the type of reform effort implemented, teachers with an inadequate understanding of mathematics or a misunderstanding of mathematical concepts will compromise student learning and the goals of the reform. For example, innovative middle school curriculum materials highlight problem-solving strategies such as making a table when studying algebraic relationships (Lappan et al. 1998a; Romberg et al. 1999). To be effective, these approaches require teachers to understand the distinction between "making a table," as an end in and of itself, and constructing a proof of an algebraic relationship. Knowing how and when to utilize a table to demonstrate an algorithm is important, but one must be vigilant to avoid misleading middle school learners to believe that the construction of tables on a relatively small number of cases generalizes to all cases and, therefore, substitutes for "proof."

Response Type 1 is "categorized by misconceptions, limited understanding, or reliance upon concrete examples." Response Type 2 is "characterized as communicating a basic understanding of algebraic concept(s)." Response Type 3 is characterized as a movement toward generalization. Response Type 4 reveals "additional insight and alternative solutions" (Scher, Curcio, & Weinberg, 2004, p. 2).

The four categories obtained for each of the three scenarios are described in Tables 1, 2, and 3.

DISCUSSION

Scenario Task A: Seating Capacity

Response Type 1 in Table 1 considers two aspects of the seating problem: the algebra and its underlying reasoning. The teacher observes that the group who answered "$4 + 2n$" reasoned correctly (i.e., there are four people at the first table and the number of people increases by two for each additional table) but faltered when translating their counting strategy into an equivalent algebraic representation. Creating a table of n values, as suggested, has the potential to uncover the nature of the algebraic mistake (specifically, a column of $4 + 2n$ values contains the same numbers as $2n + 2$, shifted up one row). Yet, the response makes no attempt to unravel the algebraic inconsistency.

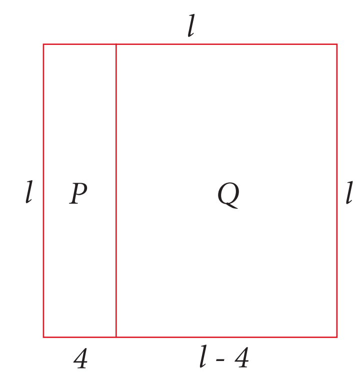
Both Response Type 2 and Response Type 3 reflect an understanding that the expression $4 + 2n$ overcounts the number of tables by one. Response Type 2 substitutes $n + 1$ in place of n to convert the correct 2n + 2 answer into the incorrect 4 + 2n. Response Type 3 operates in reverse — it replaces n by $n – 1$ to convert $4 + 2n$ into the correct $2n + 2$. While both methods have mathematical merit, Response Type 3 seems, on a pedagogical response type, more likely to aid the faltering group.

Response Type 4, in addition to explaining the algebra, offers an entirely different line of reasoning that leads to the same algebraic answer. Note that nowhere in our

TABLE 1:
Response Types for Seating Capacity Scenario Task

RESPONSE TYPE 1	RESPONSE TYPE 2	RESPONSE TYPE 3	RESPONSE TYPE 4
It is helpful to create a table and see what type of patterns appear. You may want to "guess" at an answer beforehand (such as $2n + 2$ or $4 + 2n$) and see if it makes sense based upon the results in the table. Does $2n + 2$ make sense? for $n = 1$, $2(1) + 2 = 4$ yes for $n = 2$, $2(2) + 2 = 6$ yes for $n = 3$, $2(3) + 2 = 8$ yes for $n = 4$, $2(4) + 2 = 10$ yes Seems to make sense. Does 4 + 2n make sense? For $n = 1$, $4 + 2(1) = 6$. Already doesn't work. This is a good way to use visuals and an actual simulation (i.e., draw n tables and actually count the number of people) in order to illustrate a property. The second group's reasoning is correct (i.e., there are 4 people at one table and it increases by 2 every time a table is added), but the conclusion doesn't reflect this reasoning. Going back to the table and checking against concrete numbers may help clear this up.	I would have them set up a table so it becomes easier to see a pattern and arrive at a formula. # of tables | total # of people that can be seated 1 | 4 2 | 6 3 | 8 4 | 10 n | $2n + 2$ To address the 2n + 4, show that when the number of tables is 1, 2n + 4 gives 6, and only 4 people can be seated at 1 table. The group arrived at this number by finding the next number after n on the table. E.g., # of tables | # of people seated n | $2n + 2$ $n + 1$ | $2(n + 1) + 2 = 2n + 4$	First look at how the answers are similar (both have $2n$ + something) and how they differ (the number added is 2 vs. 4). Also have groups describe what each piece of their formula stands for. The $2n + 2$ group was better at this: 2 —> number of long sides n —> number of people sitting at long side +2 —> 2 people sit on the end. If the 2nd group sees this done by the first group, they might discover their own error, which was that n represented additional tables. The challenge would then be how can the second group's formula be adapted if we go back to the original problem, which states that n is the number of tables, and how do we show it is equivalent to the first group's? $4 + 2(n - 1) = 2n + 2$ n = # of tables $n - 1$ = # of additional tables	This answer contained all elements of a Response Type 3 response as well as the following: The first group is making the right point by the way they analyze the problem. They can try to solve it another way like: If there are n tables with 4 people seated on each one, how many people will lose their seats if these tables are placed side by side? The class can then see whether the new answer matches their original answer. $4n - 2(n - 1) = 2n + 2$

TABLE 2:
Response Types for Perimeter versus Area Scenario Task

RESPONSE TYPE 1	RESPONSE TYPE 2	RESPONSE TYPE 3	RESPONSE TYPE 4
The pattern here is that the sum of the side length of the square and the perimeter equals the area. Let's say *A* = area *P* = perimeter *S* = side length of square The pattern proves that $A = P + S$. 1 = 4 + (1 x –3) 1 = 4 + –3 1 = 1 4 = 8 + (2 x –2) 4 = 8 + –4 4 = 4	I would explain why that relationship exists and explain that the student is not doing anything different than the rest of the class, only arranging the numbers differently. That is, generally area is found by squaring the side length. Here, we are doing that in a different way. Example: side length = 5 x 1 perimeter = 20 (= 5 x 4) area = 25 Generally, $5^2 = 25$. Here: (5 x 1) + (20) = (5 x 1) + (5 x 4) Factor out a 5 to get: $5(1 + 4) = 5 \times 5 = 5^2$, and actually the same as we are used to.	I would have students determine, given side *s*, the area and perimeter of a square algebraically. They would find that the area of a square is s^2 and the perimeter is 4*s* (if they do not previously know this). Then, $sx + 4s = s^2$ What must x be? Well, x must have an s so that the left side will have an s^2 term. Does $s \cdot s + 4s = s^2 + 4s = s^2$? No. So given that $s \cdot \text{something} + 4s = s^2$, and there must be at least an s, what can I multiply by *s* to get rid of 4*s*? The answer is –4. So $s(s - 4) + 4s = s^2$ by the distributive property.	This answer contained all elements of a Response Type 3 response as well as the following: Let *l* = side length of the square. We can look at it in a geometrical way:  The area of P (= 4*l*) is equal to the perimeter of the square. Good observation! The relationship does exist.

problem statement did we require an alternative strategy. But the inclusion of one in Response Type 4 demonstrates an algebraic flexibility.

Scenario Task B: Perimeter versus Area

Response Type 1 in Table 2 incorrectly states that area equals perimeter plus side length and claims that numerical data alone "proves" the algebraic relationship. Response Type 2 also focuses on numbers, but with a difference: here, the explanation skillfully manipulates the term (5 x 1) + 20 to show its equivalence to 52. The work is grounded in one specific example, but the manipulations show an understanding of numbers extending beyond calculation to more purposeful pattern finding. Only for Response Type 3 does the explanation deliver a generalized algebraic proof.

Response Type 4 includes a geometric interpretation of the underlying algebra. The work is notable, too, for including the short message, "Good observation!" While Response Type 2 states that the student's discovery is not "…anything different than the rest of the class," Response Type 4 displays a mathematical appreciation of the insight's uniqueness.

Scenario Task C: Binomial Expansion

Response Type 1 in Table 3 begins promisingly by proposing numerical substitution as a way to demonstrate the inequality of $(a + b)^2$ and $a^2 + b^2$. Nearly every response to this item, regardless of response type, included this concrete approach. It remained to establish the correct identity.

Response Type 1 offers the "FOIL" mnemonic, a rule-based method unlikely to promote conceptual understanding. Response Type 2 relates the expansion of $(a + b)^2$ to the process of multiplying two-digit numbers—a concrete link. It is unclear, however, whether teachers' knowledge of multiplication itself rises above an algorithmic understanding.

Response Type 3 stands apart from the previous answers by taking note of the context provided in our classroom scenario. Since the scenario states that an algebraic approach to the binomial expansion had not proven effective, the respondent gives a geometric representation of $(a + b)^2$ illustrating clearly the origins of the "$2ab$" term.

Most current algebra texts feature Response Type 3 ideas as a way to visualize $(a + b)^2$ (Bellman, Bragg, Chapin, Gardella, Hall, Handlin, & Manfre, 2001; Lappan et al., 1998c; McConnell, Brown, Usiskin, Senk, Widerski, Anderson, Eddins, Feldman, Flanders, Hackworth, Hirschhorn, Polonsky, Sachs, & Woodward, 1998). By contrast, Response Type 4 includes a mathematical connection that is, to the best of our knowledge, original. If $(a + b)^2$ did equal $a^2 + b^2$, then a, b, and $a + b$ may be viewed as the bases and hypotenuse of a right triangle. Yet in any triangle, the sum of two sides is always greater than the third. The equality cannot hold.

Recommendations for Professional Development

In all three scenarios, Response Type 1 relies almost exclusively on numerical data. The responses step back from algebra, using numerical substitution as a spot-check of a conjecture's viability. Response Type 1 in Table 1, for example, concludes from an inspection of tabular data that the algebraic statement $2n + 2$ "seems to make sense." Yet nowhere in the response does the teacher move beyond the suggestiveness of data to the conclusiveness of algebraic reasoning.

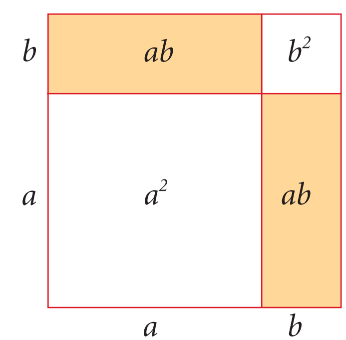
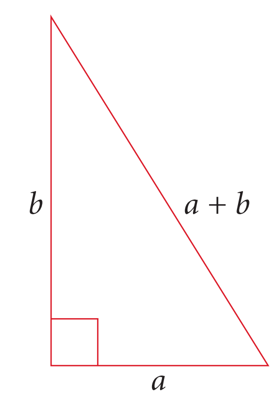
It may be possible that Response Type 1 teachers were thinking about their students when they responded to the task. As such, the teachers' reliance on numerical data may say more about the ability of their students than the teacher's own knowledge of algebra. It is suggested that teachers indicate and discuss the types of classes they teach and describe their students prior to completing the tasks, or during the completion of the tasks.

Many current algebra curricula feature tabular data (Lappan et al.1998; McConnell et al. 1998), but only as a first step towards greater generality. Response Type 1 teachers favor this approach because of its concreteness, but remain uncomfortable with the transition to algebraic representation. These teachers need help moving from concrete examples to generalizations—experiencing the "power" of algebra. As a start, teachers in a professional development workshop could answer any of our three scenario tasks and then examine the corresponding table of responses to discuss what seems to differentiate Response Type 1 from, say, Response Type 2, and then determine where each of their current thinking fits in relation to the given categories.

Describing the qualities of Response Types 3 and 4 requires some care. Certainly, these answers display a greater facility with algebraic symbolism. Facile symbol manipulation alone, though, does not guarantee algebraic

TABLE 3:

Response Types for Binomial Expansion Scenario Task

RESPONSE TYPE 1	RESPONSE TYPE 2	RESPONSE TYPE 3	RESPONSE TYPE 4
It is often useful to substitute numbers in place of letters to "check" your final answer. Explain the FOIL method of multiplying **F**irst numbers, **O**utside numbers, **I**nside numbers, and **L**ast numbers to get $a^2 + 2ab + b^2$. If some students still are not convinced that distributing the "squared" doesn't work, substitute in actual numbers: $(3 + 4)^2 \neq (32 + 42)$. Why? We can go through the FOIL process or simply add 3 + 4 to get 72 = 49 (which is concrete…kids won't argue that). Now distribute the square to get 32 + 42 = 9 + 16 = 25. $25 \neq 49$. Kids will clearly understand an actual numerical example even if they don't automatically think "FOIL."	The multiplication of two numbers involves more than a multiplication of the first terms and the second terms. It involves multiplying each digit in one number by all the digits in the other number. For example, in 45 x 10, we multiply the 0 by 5 and by the 4. A zero goes in as a placeholder, and then we multiply the 1 by the 5 and the 4. Doing this for $(a + b)^2$ we get: $a + b$ $*\ a + b$ $ab + b^2$ $a^2 + ab \quad 0$ $a^2 + 2ab + b^2$ If this gives students trouble, I'd like to break up my first example and then follow the same procedure. 45 x 10 = 40 + 5 * 5 + 5 200 + 25 200 + 25 + 0 200+ 225 + 25 = 450	Draw the following:  Show the side lengths as $a + b$. Ask what the area of the square would be (if necessary, ask how to find the area of a square before asking the area of THIS square). With students, fill in the dimensions and areas of the four pieces, possibly using different colors for the different-sized pieces. Add the pieces together to get the area —> $a^2 + 2ab + b^2$. If students are still unsure, or prefer working with numbers, have students choose numbers for a and b and test $(a + b)^2 = a^2 + b^2$. Perhaps challenge the class to find an a and b that will make this true. After a couple of tries, they may say it's impossible.	This answer contained all elements of a Response Type 3 response as well as the following: If $(a + b)^2 = a^2 + b^2$, then a, b, and $a + b$ are 3 sides of a right triangle:  Of course not! (Why not?)

maturity. The responses point to other, more subtle qualities that help to describe accomplished algebraic thinking. These are as follows:

1. Recognizing student work (at an arithmetic level) that makes unexpected connections to algebra;
2. Spotting correct reasoning among faulty algebra; and
3. Uncovering connections between algebra and geometry.

This list suggests those areas of professional development that may benefit teachers who are comfortable with algebraic manipulation, but are not facile in connecting their knowledge to the more roughly-hewn reasoning of their students. These teachers need practice in recognizing the germ of a good algebraic idea in approaches that are neither typical nor entirely accurate. Such teachers could also benefit from studying geometric arguments that illuminate the meaning behind algebraic statements. Studying the Response Types 3 and 4 in our tables is a first step in that direction.

Closing Comments

As this exploratory study makes clear, an approach that employs the use of scenario tasks based on actual classroom practice has the potential for eliciting a wide range of responses that may be systematically linked to the ways in which teachers think about and formulate their own approaches to presenting mathematical content in the classroom. By incorporating such tasks in professional development, and encouraging teachers in such sessions to reflect upon the type of responses they are most likely to produce in the classroom given a particular scenario task, we are providing them with an opportunity to evaluate their responses in comparison to others and to become more flexible in their mathematical thinking in a non-threatening and supportive setting. Furthermore, because scenario tasks are content specific (e.g., algebra-based), they are best suited for professional development sessions structured by mathematical topic and grade level.

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